Partnership Dissolution with Cash-Constrained Agents

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Abstract

When partnerships come to an end, partners must find a way to efficiently reallocate the commonly owned assets to those who value them the most. This requires that the aforementioned members possess enough financial resources to buy out others' shares. I investigate ex post efficient partnership dissolution when agents are ex post cash constrained. I derive necessary and sufficient conditions for ex post efficient partnership dissolution with Bayesian (resp. dominant strategy) incentive compatible, interim individually rational, ex post (resp. ex ante) budget balanced and ex post cash-constrained mechanisms. Ex post efficient dissolution is more likely to be feasible when agents with low (resp. large) cash resources own more (resp. less) initial ownership rights. Furthermore, I propose a simple auction to implement the optimal mechanism. Finally, I investigate second-best mechanisms when cash constraints are such that ex post efficient dissolution is not attainable.

Keywords: *Mechanism design, Partnership, Ex post cash constraints, Property rights theory.*

JEL Classification: D02, D23, D40, D44, D82, C72.

1. INTRODUCTION

The Coase theorem stipulates that when transaction costs are sufficiently low, negotiations will always lead to an efficient outcome regardless of the initial allocation of ownership rights. Unfortunately, however, assuming something as simple as asymmetric information means that those transaction costs are no longer negligible. This is illustrated by the major contribution of Myerson and Satterthwaite (1983) who consider trade between a seller (the owner) and a buyer with bilateral asymmetric information. Their striking result is that no mechanism can achieve

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ex post efficient trade. Cramton, Gibbons and Klemperer (1987, henceforth CGK) show that the Myerson and Satterthwaite (1983) impossibility result becomes a possibility result if agents initially own equal (or close to equal) shares. The economic insight is that ownership rights give agents bargaining power in negotiations. When this bargaining power is excessive - the seller has “monopoly” power over the good in Myerson and Satterthwaite (1983) – negotiations fail. Equal-share ownership, on the contrary, is enough to curb the bargaining power of each agent and restore trade efficiency.

However, allocating ownership rights not only allocates bargaining power but it also determines the volume of trade. If efficiency requires one agent to buyout the shares of $(n-1)$ other agents, equal-share ownership will require the transfer of a fraction $(n-1)/n$ of all shares. It therefore means that the buyer will have to assume large cash payments to compensate the sellers. If some agents are financially constrained, such payments may be unfeasible when traded volumes are excessive and equal-share ownership might not be desirable anymore.

In this paper, I consider limited cash resources as another source of inefficiency in those trading problems. I investigate how trading mechanisms should be constructed and which ownership structures allow for efficient trade when agents are cash constrained. Formally, I build on the partnership dissolution model, first initiated by CGK, in which I consider agents with (possibly asymmetric) cash resources. In this framework, each partner initially owns shares of common assets. Dissolution simply means that following some event (disagreement, natural termination, bankruptcy), the commonly owned assets must be reallocated, that is, each partner will buy or sell their share to others. The framework therefore applies to various economic problems such as divorce, inheritance, termination of joint-ventures, privatizations. Examples of applications are covered in more detail below.

Although the presence of cash – or budget – constraints seems to be a reasonable claim, little is still known about the design of trading mechanisms with cash-constrained agents. In auction design, earlier contributions of Laffont and Robert (1996), Che and Gale (1998, 2006) and Maskin (2000) have investigated cash constraints in standard auction settings. More recently, several authors have recognized limited cash resources as one of the gaps that limits the implementation of theory into practice.[1] In partnership problems, limited cash resources are directly linked to the initial distribution of ownership. Buying out a partner with large initial entitlements requires the ability to raise enough money to compensate them. In privatization of public-private partnerships or in spectrum allocations, cash constraints matter as the value of traded assets is often worth millions.[2] In divorce, inheritance or joint-venture problems, it is likely that agents have limited access to credit markets. Cash resources also create some

[1] See, among others, Dobzinsky, Lavi and Nisan (2012), Bichler and Goeree (2017), Carbajal and Mu'Alem (2018) and Baisa (2018). Two other important recent contributions to auction design with cash constraints are Pai and Vohra (2014) and Boulatov and Severinov (2019).

[2] Cramton (1995) thinks that cash constraints have played a major role in the Nationwide Narrowband PCS Auction in 1994. Some bidders had likely dropped out from the auction because of limited resources although they had large valuations for the good.

kind of bargaining power: Partners with large cash resources could take advantage of very cash-constrained partners.

Applications. I now present some applications that can be addressed within the partnership dissolution framework.

(i) Divorce, inheritance: Marriage or civil union represent, among other aspects, the joint ownership of some assets and the pursuance of a common goal. If dissolution occurs, *i.e.* divorce, the partners ought to agree on the reallocation of the family home, cars, and other possessions. Limited financial resources, especially compared to the market value of the family home, may complicate the process of finding an agreement on who should be the final owner. Alternatively, the assets to be traded may have more sentimental value than market value (*e.g.* the inherited childhood home) so that it may be difficult to use it as a collateral to borrow from a bank. Along the same lines, heirs of the deceased's property (real estate, business, debts) may want to reallocate the inheritance differently to what they had initially been entitled by the testament.

(ii) Joint ventures: Business associates, joint ventures or venture capital firms are often governed by partnership law. For instance, in biotechnology and high-technology sectors, it is common that strong-potential young firms with low financial resources decide to rely on alliances with larger firms to compensate for the lack of complementary assets and liquidity (see Aghion and Tirole (1994) ; Lerner and Merges (1998) ; Aghion, Bolton and Tirole (2004)). Interestingly, Aghion, Bolton and Tirole (2004) point out that not only can dissolution be triggered by dispute or unsuccessful results but it may also be due to the very nature of this form of partnership. Indeed, Aghion, Bolton and Tirole (2004) report that those partnerships are generally temporary by nature and the young firms eventually seek other sources of funding requiring an exit from the partnership. As recognized by both Aghion and Tirole (1994) and Lerner and Merges (1998), the presence of cash constraints for the young firm generates inefficiencies in investment decisions as well as in allocation of ownership.

(iii) Bankruptcy procedures: Wolfstetter (2002) mentions that some bankruptcy procedures can be seen as a partnership dissolution problem. One example of a bankruptcy procedure, a cash auction, is given by Aghion, Hart and Moore (1994): All remaining assets of the bankrupt firm are simply sold in an auction to the highest bidder. Some bidders may be former owners (with positive ownership rights in the firm) and other may be outsiders (with null ownership rights). Aghion, Hart and Moore (1994) believe that a cash auction would be the "ideal bankruptcy procedure" (p. 855) in the absence of the difficulty to raise enough cash to buy the firm at its true value. They argue that cash constraints will likely result in a lack of competition in the auction and the firm would then be sold at a low price. This stresses the importance of designing mechanisms that directly take cash constraints into consideration.

(iv) Land reallocation: Land reallocation problems may also be challenged by the partnership

framework. Che and Cho (2011) report the inefficiencies of the initial land allocation in the Oklahoma Land Rush in the late 19th century. Therefore, reallocation of those lands required to take into consideration the initial ownership structure induced by the first allocation. More recently, Loertscher and Wasser (2019) mention that land reallocation will be a major challenge in China. Starting in 1978, several reforms occurred in China to give household land use rights to farmers and then secure household land transfer rights (2002, 2007, 2008) while the land is still collectively owned by villages.[3] Participation by farmers in reallocating their land must then be voluntary and compensated by monetary transfers. The State Council of China believes that agricultural modernization in China will occur through the reallocation of the use of land from traditional farmers to a new generation of farmers (professional farmers or dragonhead enterprises; see Zhang (2018)). Traditional farmlands are considered too small and inefficient and reallocating them to larger and more skilled producers would help the modernization of agriculture in China.

CONTRIBUTION. The main feature of a partnership model is that participation constraints in the dissolution mechanism depend on the initial allocation of ownership rights among partners. Partners with relatively more initial shares have a higher claim and their participation in the mechanism is harder to ensure, making the initial ownership structure a determinant condition for optimal dissolution. In their seminal paper, CGK answer this problem by designing ex post efficient, Bayesian incentive compatible, interim individually rational and ex post budget balanced mechanisms. Their main finding consists in characterizing the initial allocations of property rights among partners that allow for ex post efficient dissolution. They show that equal-share partnerships can always be ex post efficiently dissolved whereas partnerships with excessive concentration of ownership are less likely to be dissolved efficiently. In particular, in a two-agent partnership with extreme ownership – one agent owns the whole asset –, their model collapses to the one of Myerson and Satterthwaite (1983) which proves the impossibility of ex post efficient dissolution in extreme ownership partnerships.

One of the main simplifying assumptions in CGK, however, is that all partners are always endowed with enough money so that they are able to pay for the monetary transfers proposed by the dissolution mechanism. While this assumption may seem innocuous in many contexts, I argue that it is not the case in the partnership dissolution problem. First, one of the purposes of a partnership generally consists in sharing the burden of acquiring costly assets such as firm premises, industrial equipment, computer hardware and software for businesses, or real estate, cars and household appliances for a couple. In some business partnerships, one agent provides the physical capital while the other one provides the human capital.[4] It is also possible that dissolution occurs precisely when partners face financial difficulties. That is, some partnerships

[3]See Ma *et al.* (2013) for a detailed chronology of land reforms in China.

[4]For instance, Landeo and Spier (2014) cite the *Haley v. Talcott* case. In, 2001, the two of them started a restaurant in Delaware, Talcott provided the capital while Haley was supposed to manage the restaurant without salary for the first year.

may be, by essence, constituted by cash-constrained agents.[5] Second, from both the theoretical and practical point of view, introducing cash constraints in the partnership dissolution model of CGK requires nontrivial changes in the design of monetary transfers. For instance, ex post monetary transfers in cash-constrained mechanisms must be bounded and must also satisfy the exact same conditions as in CGK at the interim level; namely incentive compatibility and individual rationality. Constructing those transfers therefore requires to find ex post transfers with a lower range but with the exact same marginals when projected at the interim level. In CGK, and in many other works on partnership dissolution, the absence of cash constraints is implicitly used and greatly simplifies the construction of dissolution mechanisms but they may also induce unreasonably high transfers for some states of the world.

In this paper, I first investigate the possibility of ex post efficient partnership dissolution when partners are ex post cash constrained, that is, when they have an upper bound on the payments they can make to other partners to buy out their shares. I derive necessary and sufficient conditions for ex post efficient partnership dissolution with Bayesian (resp. dominant strategy) incentive compatible, interim individually rational, ex post (resp. ex ante) budget balanced and ex post cash-constrained mechanisms. While the necessary and sufficient conditions for ex post efficient dissolution end up being quite a natural generalization of the results of CGK, the construction of the mechanism transfer function requires some extra work. I fully characterize these conditions and I show that the equal-share partnership is no longer the initial ownership structure that ensures feasibility of ex post efficient dissolution. Instead, partners who are initially relatively more (resp. less) cash-constrained than others must receive relatively more (resp. less) initial ownership rights. Intuitively, the more cash-constrained a partner is, the higher the utility they would receive in the mechanism (as they cannot be asked to pay much but they could still receive the asset) relatively to less cash-constrained partners. Thus, a very cash-constrained partner with few initial ownership rights will always be willing to participate (low maximal monetary transfers and low utility if they refuse the mechanism). It follows that giving more initial ownership rights to these cash-constrained partners does not change their participation decision but it implies that other less cash-constrained partners receive less initial ownership rights, which reduces their claim. This result sheds light on a new link between liquid and illiquid assets in partnership regardless of prior investment decisions. It is worth noting that the asymmetry in cash constraints among partners drives the result that *optimal* initial ownership rights must also be asymmetric. If partners all have the same exact cash resources, the equal-share partnership is still *optimal* as in CGK.

Focusing on equal-share and equal-cash-resources partnerships, I characterize the minimal amount of cash each partner must hold so that ex post efficient dissolution is achievable. Interestingly, the minimal amount of cash resources each partner must possess is increasing in the number of partners and converges to the maximal possible value of the asset when the

[5]Some companies are specialized in providing short-term financial resources to partners facing a dissolution, see the *Shotgun fund*, for instance.

number of partners becomes large. For instance, if a four-agent partnership with equal-share owns an asset worth 1 million and valuations are uniformly distributed on the unit interval, each agent must possess 0.8 million for ex post efficient dissolution to be possible. This result stresses the importance of taking cash constraints into account as they appear to be quite restrictive.

Interestingly, the standard equivalence theorem between Bayesian and dominant strategy implementation is robust to the addition of cash constraints. That is, any ex post efficient, interim individually rational and ex post cash-constrained mechanism can be implemented in dominant strategies with ex ante budget balance or with bayesian incentive compatibility with ex post budget balance. Moreover, transfers in both cases can be interim equivalent for all agents, that is, one can ensure the same interim utilities for all agents.[6] It also appears that there is an equivalence between ex post cash constraints and interim cash constraints. More precisely, I show that relaxing cash constraints from the ex post level to the interim level does not weaken the conditions for ex post efficient dissolution.

As the general mechanism design formulation is often difficult to apply, I propose a simple bidding game that implements the ex post efficient dissolution mechanisms. In this bidding game, partners receive/pay an upfront transfer and then simply submit bids in an all-pay auction. It is constructed such that the bidding strategy of a partner is increasing in their valuation so that the highest bidder (the winner) is also the partner with the highest valuation. The upfront payment ensures interim individual rationality, budget balance and cash constraints. This bidding game can replace the one proposed by CGK, which fails to satisfy cash constraints when some or all partners have low cash resources.

Finally, I investigate second-best mechanisms to characterize the optimal allocation of final ownership rights when cash constraints are such that the first-best allocation is not attainable. First, I characterize all incentive compatible, interim individually rational and cash-constrained mechanisms for any possible allocation rule. Second, building on Lu and Robert (2001), Loertscher and Wasser (2019) and Boulatov and Severinov (2018), I show that solving the problem of maximizing a weighted sum of ex ante gains from trade and the revenue collected on agents requires allocating the asset to the agent(s) with the highest *ironed virtual valuation*. As in Lu and Robert (2001), Loertscher and Wasser (2019), the solution of maximizing ex ante gains from trade subject to incentive compatibility, interim individually rationality, cash constraints and budget balance is simply a particular solution to the previous problem for a specific weight.

Literature Review. Several contributions related to cash constraints can be found in the auction literature. Laffont and Robert (1996) characterize optimal auctions under independent

[6]This equivalence has first been studied by Mookherjee and Reichelstein (1992) and Makowski and Mezzetti (1994) for the same classes of mechanisms but without cash constraints. Kosmopoulou (1999) proves the equivalence when replacing interim individual rationality constraints by ex post ones.

valuations and symmetric cash-constrained agents. Maskin (2000) also examines constrained efficiency in auctions with symmetrical cash constraints. Malakhov and Vohra (2008) restrict the analysis to two-agent problems but assume only one agent is cash-constrained. More recently, Boulatov and Severinov (2018) propose a complete characterization of optimal auctions with asymmetrically cash-constrained bidders. However, adding cash constraints in a partnership problem is different than in the auction settings, namely, the budget balance requirement in partnership problems ties all partners monetary transfers together which makes the construction of those monetary transfers a challenge in itself.

Another strand of the literature considers agents whose valuations *and* cash resources are private information. Che and Gale (1998, 2006) compare the performances of standard auctions in that case. Che and Gale (2000) study the optimal pricing of a seller facing a cash-constrained buyer who has private information over their valuation and "budget". Pai and Vohra (2001) derive optimal auctions when both valuations and budget are private information. Assuming that cash resources are private information is an interesting feature, however, as it creates a multidimensional incentive compatibility problem, I will consider only commonly known budget in the present paper.

Finally, it is worth noting that ex post cash constraints are quite different from the ex post individual rationality requirement that has been extensively studied in the literature with Gresik (1991), Makowski and Mezzeti (1994), Kosmopolou (1999) and Galavotti, Muto and Oyama (2011) among others. The ex post individual rationality constraints, sometimes called "budget constraints" or "ex post regret-free", require that agents do not have negative net utility ex post. On the contrary, *pure* ex post cash constraints ignore the utility a partner derives from the share of ownership they receive in the dissolution mechanism. In a mechanism where a good is traded among several agents, the ex post participation constraints are generally harder to satisfy for the agents who receive nothing as they do not enjoy utility from consumption. As for the ex post cash constraints, they are generally harder to satisfy for the agent who receives the highest quantity of the good, as they must pay higher prices (due the monotonicity of the allocation rule under incentive compatibility) and that the utility generated from the consumption is ignored. In other words, ex post cash constraints assume that the ability to pay of an agent in a mechanism cannot be contingent on what they receive in the mechanism.

Organization of the paper. In Section 2, I present the theoretical framework for studying partnership dissolution mechanisms. Section 3 gives necessary and sufficient conditions to achieve ex post efficiency with cash-constrained agents. Section 4 provides characterization results of these conditions. Section 5 proposes a bidding game that replicates the mechanism through an all-pay auction. Section 6 second-best mechanisms. Finally, Section 7 proposes some extensions.

2. THEORETICAL FRAMEWORK

Consider a finite number of risk-neutral agents $n \geqslant 2$ indexed by $i \in N := \{1, \dots, n\}$. Each agent $i \in N$ initially owns a share $r_i \in [0,1]$ of a perfectly divisible asset, where $\sum_{i \in N} r_i = 1$.[7] Each agent $i \in N$ has private information over their valuation v_i for the asset. Valuations are independently distributed according to a commonly known cumulative distribution function F with support $V := [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$ and density function f.[8] Further assume that F is absolutely continuous. Let $v := (v_1, \dots, v_n) \in V^n$ and $r := (r_1, \dots, r_n) \in \Delta^{n-1}$ denote the vectors of valuations and initial shares, respectively. This defines the standard partnership framework as first studied by CGK.

The additional assumption I require concerns agents' cash resources. Each agent $i \in N$ is endowed with some amount of cash $l_i \in \mathbb{R}_+$. This amount represents the upper bound on payments that agent i can be requested to make in the mechanism. The source of these cash constraints is not explicitly modeled here and each l_i is considered to be exogenously determined and publicly known at the beginning of the game.[9] Let $l = (l_1, \dots, l_n) \in \mathbb{R}^n_+$ denote the vector of agents' cash resources.

Dissolving the partnership consists in reallocating the commonly owned asset to the agents who value it the most. As valuations for the asset are private information to the agents, dissolution requires to make them reveal their valuations. By the Revelation Principle, the analysis can be restricted to the search of direct revealing mechanisms in which each agent's optimal strategy consists in *truthfully* revealing their valuation. Such mechanisms will be refer to as *dissolution mechanisms*.

In a dissolution mechanism, each agent reports their valuation v_i and then receives an allocation of the asset $s_i(v)$ and a monetary transfer $t_i(v)$, both depending on the vector of all reports $v \in V^n$. Let $s(v) := (s_1(v), \dots, s_n(v))$ denote the *allocation rule*, where $s_i : V^n \to [0,1]$ such that $\sum_{i \in N} s_i(v) = 1$ for all $v \in V^n$, and $t(v) := (t_1(v), \dots, t_n(v))$ denote the *transfer rule* where $t_i : V^n \to \mathbb{R}$. By convention, the couple (s,t) represents a dissolution mechanism implementing allocation rule s with transfers t.

ILLUSTRATING EXAMPLE. To illustrate the theoretical framework, consider the following example.[10] A pharmaceutical firm, say partner 1, and a R&D firm, say partner 2, decide to form a partnership to develop a new pharmaceutical drug. Initial ownership, r_1 and r_2, represent claims on final output, that is, shares that each firm has the right to retain on the final value of

[7]The requirement that $\sum_{i \in N} r_i = 1$ is not necessary to derive the condition under which ex post efficient dissolution is feasible. I chose to impose it to fit the partnership dissolution story of CGK. In Section 7, I provide an extension and examples in which property rights can take many other forms.

[8]In Section 7, I show that the main results can easily be extended to asymmetric distributions of independent valuations.

[9]Those limited cash resources can be the result of different financial situations of the agent after considering their personal wealth, borrowing capacities, debts or limited liability.

[10]The example is inspired by Minehart and Neeman (1999).

the partnership. It is common that pharmaceutical firms own shares of small R&D firms to whom they provide capital and liquidity. Valuations, v_1 and v_2, represent perceived potential cash flows from exploiting the drug. Finally, l_1 and l_2 corresponds to each firm's financial resources (cash holdings, borrowing capacities). Once the drug has been developed, the two firms negotiate the rights to exploit it. Either the pharmaceutical buys out the R&D firm and sell the drug on the market, or the R&D firm obtains full ownership and try to sell it to another pharmaceutical company. The dissolution mechanism will (i) make each firm truthfully report their valuation so that it is possible to allocate the drug to the one with the highest valuation; (ii) determine associated monetary transfers to compensate the partner that relinquishes their claim on the product.

UTILITY. The utility function of agent i is assumed to be linear in ownership shares and separable in money. Hence, agent i has utility $v_i \alpha_i + \beta_i$ when they own a share of the asset α_i and has an amount of money of β_i. Therefore, participation in a dissolution mechanism (s, t) gives agent i utility (net of initial ownership rights):

$$u_i(v) := v_i(s_i(v) - r_i) + t_i(v).$$

By convention, when a function is evaluated at a vector (v_i, v_{-i}) it is implicitly assumed that the argument are still ordered by the agents' indices, where $v_{-i} \in V^{n-1}$ is the vector of all agents' valuations except the one of agent i. For instance, $s_i(v_i, v_{-i}) = s_i(v_1, v_2, \ldots, v_n)$.

As valuations are private information, each agent considers their interim utility, *i.e.* their utility averaging over all other agents' valuations (given that they all report truthfully). Let $U_i(v_i)$ be agent i's interim utility, that is,

$$U_i(v_i) := v_i(S_i(v_i) - r_i) + T_i(v_i),$$

where $S_i(v_i) := \mathbb{E}_{-i} s_i(v)$, $T_i(v_i) := \mathbb{E}_{-i} t_i(v)$ and where $\mathbb{E}_{-i}$ is the expectation operator over all valuations except v_i.

INCENTIVE COMPATIBILITY. The first property required on a dissolution mechanism is that it induces information revelation, thereafter called *incentive compatibility*. Two standard notions of incentive compatibility will be considered separately: (i) *interim* incentive compatibility (IIC), and (ii) *ex post* incentive compatibility (EPIC). Formally, IIC and EPIC are defined as follows.

Definition 1 *A dissolution mechanism* (s, t) *is interim incentive compatible (IIC) if for all* $i \in N$, $v_i \in V$ *and* $\tilde{v}_i \in V$,

$$U_i(v_i) \geqslant v_i(S_i(\tilde{v}_i) - r_i) + T_i(\tilde{v}_i).$$

Definition 2 *A dissolution mechanism* (s, t) *is ex post incentive compatible (EPIC) if for all* $i \in N$, $v_i \in V$, $\tilde{v}_i \in V$ *and* $v_{-i} \in V^{n-1}$,

$$u_i(v_i, v_{-i}) \geqslant v_i(s_i(\tilde{v}_i, v_{-i}) - r_i) + t_i(\tilde{v}_i, v_{-i}).$$

Definitions 1 (resp. 2) simply defines dissolution mechanisms (s,t) such that truth-telling is a Bayesian Nash (resp. dominant strategy) equilibrium. Notice that if a dissolution mechanism (s,t) is EPIC then it is also IIC, but the reverse is not necessarily true. Both IIC and EPIC will be investigated in the first-best analysis whereas I will restrict to IIC for the analysis of second-best mechanisms.

Individual Rationality. The second property of a dissolution mechanism is that participation is voluntary. Following CGK, I require that dissolution mechanisms are interim individual rational (IIR). Given that utilities are defined net of the initial ownership shares, IR is defined as follows.

Definition 3 *A dissolution mechanism* (s,t) *is interim individually rational (IIR) if for all* $i \in N$ *and* $v_i \in V$,

$$U_i(v_i) \geqslant 0.$$

Notice that IIR depends on r_i for each agent i. The higher r_i the more difficult it is to satisfy $U_i(v_i) = v_i(S_i(v_i) - r_i) + T_i(v_i) \geqslant 0$ for a given mechanism (s,t). This is the main feature of partnership problems: The initial distribution of ownership shares has a direct impact on the feasibility of dissolution mechanisms through the constraints it imposes on each agent's minimal claim.[11]

Budget Balance. A dissolution mechanism (s,t) is said to be budget balanced when no subsidy is required to implement the allocation rule s. Two notions of budget balance are considered: (i) *ex post* budget balance (EPBB) , and (ii) *ex ante* budget balance (EABB). Formally, these two standard notions write:

Definition 4 *A dissolution mechanism* (s,t) *is ex post budget balanced (EPBB) if for all* $v \in V^N$,

$$\sum_{i \in N} t_i(v) = 0.$$

Definition 5 *A dissolution mechanism* (s,t) *is ex ante budget balanced (EABB) if*

$$\mathbb{E}\left[\sum_{i \in N} t_i(v)\right] = 0.$$

EPBB implies that for every profile of valuation $v \in V^N$, the transfers required to implement the allocation rule s cancel out between agents. EABB, however, only requires transfers to cancel out *on average*. Therefore, EPBB is, of course, a stronger requirement than EABB. As it will be shown below, the only advantage of relaxing budget balance from the ex post level

[11]See Section 7 for an extension of more general interim individual rationality constraints. I show that the framework is not limited to partnership dissolution problems and can be easily extended to study the problem of optimally allocating a good to agents with type-dependent outside options.

(EPBB) to the ex ante one (EABB) consists in being able to impose incentive compatibility at the ex post level (EPIC) rather than at the interim level (IIC).

Cash-Constrained Mechanisms. Finally, I require that dissolution mechanisms satisfy ex post cash constraints, that is, no agent can be required to pay more than their cash resources.

Definition 6 *A dissolution mechanism* (s, t) *is ex post cash-constrained (EPCC) if for all* $i \in N$ *and* $v \in V^n$,

$$t_i(v) \geqslant -l_i.$$

The cash-constrained requirement is imposed at the ex post level and therefore assumes the most extreme form of cash constraints. As it will be shown later, ex post cash constraints are equivalent to interim cash constraints so that relaxing the ex post requirement to the interim level has no benefit.

3. EX POST EFFICIENT DISSOLUTION

I start by investigating *ex post efficient dissolution mechanisms* or, equivalently, *first-best* dissolution mechanisms. I provide necessary and sufficient conditions for the existence of such mechanisms when agents are cash-constrained. Two types of dissolution mechanisms are investigated. First, I will consider dissolution mechanisms simultaneously satisfying IIC, IIR, EPBB and EPCC, thereafter referred to as *Bayesian mechanisms*. Second, I will consider dissolution mechanisms simultaneously satisfying EPIC, IIR, EABB and EPCC, thereafter referred to as *dominant strategy mechanisms*. These two classes of mechanisms have been widely studied and one of the most important result in the literature is that they are *equivalent* in various environments.[12] Not surprisingly, the equivalence between Bayesian mechanism and dominant strategy mechanisms extends to an environment with ex post cash constraints.

Ex Post Efficient Allocation Rule. Ex post efficiency requires that the allocation rule maximizes the gains from trade for every realization of valuations $v \in V^n$. Let s^* denote the ex post efficient allocation rule, it must satisfy for all $v \in V^n$

$$s^*(v) \in \arg\max_{s \in \Delta^{n-1}} \sum_{i \in N} v_i s_i(v).$$

The solution to this linear problem simply requires to allocate full ownership rights to the agent with the highest valuation. In case of tie between two or more agents (*i.e.*, they have the same valuation) assume, without loss of generality, that the agent with the lowest index is allocated the whole asset.[13] Therefore, the ex post efficient allocation rule for agent i can

[12] The word *equivalent* is stressed here as this notion has to be carefully defined and may vary across different equivalence theorems. See Manelli and Vincent (2010) on that matter.

[13] As $F(\cdot)$ is assumed to be absolutely continuous, ties occur with probability zero and thus can be ignored in the design of the ex post efficient mechanism. However, this will no longer be the case in the second-best analysis.

simply be written as

$$s_i^*(v) = \begin{cases} 1 & \text{if } \rho(v) = i \\ 0 & \text{if } \rho(v) \neq i, \end{cases} \tag{1}$$

where $\rho(v) := \min \{ j \in N \mid j \in \arg\max_i v_i \}$, so that ties are always broken in favor of the agent with the lowest index.

3.1. *Groves Mechanisms*

To derive the main condition for ex post efficient dissolution of partnerships, I rely on the methodology derived by Makowski and Mezzetti (1994). They show that every mechanism satisfying both ex post efficiency (EF) and incentive compatibility (either interim or ex post) must be a Groves mechanism, and can be fully characterized by a specific transfer function defined up to a constant.

Let $g(v) := \sum_{i \in N} v_i s_i^*(v)$ denote the maximum gains from trade at $v \in V^n$ and let the transfer function writes

$$t_i^*(v) = g(v) - v_i s_i(v) - h_i(v), \tag{2}$$

for some function $h_i : V^n \to \mathbb{R}$ and all $i \in N$. The first two terms of this function, namely $g(v) - v_i s_i(v)$, ensure that the mechanism implements the ex post efficient allocation rule by inducing revelation. The last term, $-h_i(v)$, is an arbitrary function whose purpose is to collect money back from agent without distorting their incentives to reveal information. The following proposition details the required properties of the function h_i.

Proposition 1 ***(Makowski and Mezzetti (1994))*** *The dissolution mechanism* (s^*, t^*) *is*

- *a. EF and IIC if and only if* t^* *satisfies* (2) *and* $\mathbb{E}_{-i} h_i(v_i, v_{-i}) = \mathbb{E}_{-i} h_i(v_i', v_{-i}) = H_i$ *for all* $v_i, v_i' \in V$ *and* $H_i \in \mathbb{R}$ *is a constant;*
- *b. EF and EPIC if and only if* t^* *satisfies* (2) *and* $h_i(v_i, v_{-i}) = h_i(v_i', v_{-i})$ *for all* $v_i, v_i' \in V$.

Proof. See Makowski and Mezzetti (1994) ■

I now turn to the existence of ex post efficient Bayesian and dominant strategy dissolution mechanisms.

3.2. *The Existence Condition*

Proposition 1 ensures that a dissolution mechanism (s^*, t^*) satisfies EF and IIC (resp. EPIC) if transfers follows (2) and the function $h_i(\cdot)$ is independent of v_i on average over v_{-i} (resp. independent of v_i). Therefore, investigating ex post efficient Bayesian or dominant strategy mechanisms can be done by directly imposing all other requirements (individual rationality, budget balance and cash constraints) on t^*. I restrict the presentation of the argument to ex post efficient Bayesian mechanisms. The case of dominant strategy mechanisms is almost exactly the same and therefore relegated to Appendix A.

BAYESIAN MECHANISMS. Take a dissolution mechanism (s^*, t^*) satisfying EF and IIC (*i.e.* t^* satisfies Proposition 1.a). Imposing EPBB requires $\sum_{i \in N} t_i^*(v) = 0$ for all $v \in V^n$ or, equivalently,

$$\sum_{i \in N} h_i(v) = (n-1)g(v), \text{ for all } v \in V^n. \tag{3}$$

The term $(n-1)g(v)$ can be interpreted as the ex post deficit generated by an EF-IIC mechanism. EPBB then implies that the $h_i(\cdot)$ functions are designed to absorb this deficit while satisfying the requirement of Proposition 1.a. At the ex ante stage, EPBB requires (taking expectations over all v on both sides of equation (3))

$$\sum_{i \in N} H_i = (n-1)G, \tag{4}$$

where $H_i := \mathbb{E}h_i(v) = \mathbb{E}_{-i}h_i(v)$ and $G := \mathbb{E}g(v)$.

At the same time, imposing IIR on (s^*, t^*) requires that $U_i(v_i) = v_i(S_i^*(v_i) - r_i) + T_i^*(v_i) \geqslant 0$ for all $v_i \in V$ and $i \in N$. Replacing $T_i^*(v_i) := \mathbb{E}_{-i}t_i^*(v)$ by its expression (given by taking expectations $\mathbb{E}_{-i}$ of equation (2)) gives $\mathbb{E}_{-i}g(v) - H_i - v_i r_i \geqslant 0$. Rearranging, IIR requires that

$$H_i \leqslant \mathbb{E}_{-i}g(v) - v_i r_i, \text{ for all } v_i \in V, i \in N.$$

Define,

$$C(r_i) := \inf_{v_i} \{\mathbb{E}_{-i}g(v) - v_i r_i\}. \tag{5}$$

Then IIR be can rewritten as

$$H_i \leqslant C(r_i) \text{ for all } i \in N. \tag{6}$$

Equation (6) simply defines $C(r_i)$ as the maximal amount of money that can be collected on agent i at the interim stage without violating IIR. Notice that $C(r_i)$ is a decreasing function of r_i. This reflects that initial shares provide bargaining power to their owner: Higher initial shares allows an agent to claim a larger part of the gains from trade.

Finally, imposing EPCC on (s^*, t^*) gives $t_i^*(v) = g(v) - v_i s_i^*(v) - h_i(v) \geqslant -l_i$ for all $v \in V^n$

and $i \in N$. This implies that at the interim stage (taking expectation over all v_{-i}):

$$H_i \leqslant \mathbb{E}_{-i}\left[g(v) - v_i s_i^*(v)\right] + l_i, \text{ for all } v_i \in V, i \in N.$$

Straightforward computations give that $\min_{v_i} \mathbb{E}_{-i}\left[g(v) - v_i s_i^*(v)\right] = 0$ at $v_i = \overline{v}$ and thus the above equation simply rewrites:

$$H_i \leqslant l_i, \text{ for all } v_i \in V, i \in N. \tag{7}$$

Equation (7) yields the maximal amount of money that can be collected on agent i at the interim stage due to the presence of cash constraints.

THE DISSOLUTION CONDITION. It appears that the existence of ex post efficient Bayesian dissolution mechanisms simply relies on whether it is possible to collect enough money from agents to cover the ex ante deficit $(n-1)G$ given the upper bounds $C(r_i)$ and l_i implied by IIR and EPCC, respectively. I now state the existence condition of ex post efficient Bayesian dissolution mechanisms.

Theorem 1 *An EF, IIC, IIR, EPBB and EPCC dissolution mechanism exists* ***if and only if***

$$\sum_{i \in N} \min\{C(r_i), l_i\} \geqslant (n-1)G. \tag{8}$$

Proof. (Necessity) Equations (6) and (7) are both necessary conditions for IIR and EPCC, respectively. Combining the two equations gives $H_i \leqslant \min\{C(r_i), l_i\}$ for all $i \in N$. Summing over $i \in N$ gives $\sum_{i \in N} H_i \leqslant \sum_{i \in N} \min\{C(r_i), l_i\}$. Finally, using equation (4) (implied by EPBB) yields $(n-1)G \leqslant \sum_{i \in N} \min\{C(r_i), l_i\}$ which concludes the necessity part.

(Sufficiency) To prove sufficiency, I explicitly construct a transfer function that satisfies all requirements when (8) holds. Let $t_i^C(v) = g(v) - v_i s_i^*(v) - h_i^C(v)$ where $h_i^C(v)$ is defined as follows:

$$h_i^C(v) := \frac{n-1}{n}\left[g(v) + \frac{1}{n-1}\sum_{j \neq i} s_j^*(v)\psi(v_j) - s_i^*(v)\psi(v_i)\right] + \phi_i, \tag{9}$$

where

$$\psi(v_k) := \frac{\int_{\underline{v}}^{v_k} F(x)^n dx}{F(v_k)^n}, \tag{10}$$

and $\phi_i \in \mathbb{R}$ is a constant. I further discuss the choice of the transfer function after the proof.

EF and IIC. The transfer function $t_i^C(v)$ writes as equation (2). Therefore, if $h_i^C(v)$ satisfies Proposition 1.a then the mechanism (s^*, t^C) is EF and IIC. Standard computations (see Appendix A) give

$$\mathbb{E}_{-i}h_i^C(v) = \frac{n-1}{n}G + \phi_i.$$

Hence, $\mathbb{E}_{-i}h_i^C(v)$ does not depend on v_i and satisfies Proposition 1.a. Let $H_i^C := \mathbb{E}_{-i}h_i^C(v)$. The mechanism (s^*, t^C) is EF and IIC.

EPBB. The dissolution mechanism (s^*, t^C) is EPBB if it satisfies $\sum_{i\in N} t_i^C(v) = 0$ for all $v \in V^N$. Notice that,

$$\begin{aligned}\sum_{i\in N} t_i^C(v) &= (n-1)g(v) - \frac{n-1}{n}\Big[ng(v) + \frac{1}{n-1}\sum_{i\in N}\sum_{j\neq i} s_j^*(v_j)\psi(v_j) - \sum_{i\in N} s_i^*(v)\psi(v_i)\Big] - \sum_{i\in N}\phi_i \\ &= -\sum_{i\in N}\phi_i.\end{aligned}$$

EPBB is therefore equivalent to $\sum_{i\in N}\phi_i = 0$.

IIR. Recall from equation (6) that IIR requires $H_i^C \leqslant C(r_i)$. Hence, (s^*, t^C) satisfies IIR if

$$H_i^C = \frac{n-1}{n}G + \phi_i \leqslant C(r_i), \quad \text{for all } i \in N.$$

EPCC. Finally, EPCC requires that $t_i^C(v) \geqslant -l_i$ for all $i \in N$, $v \in V^n$, or, equivalently, $\min_{v\in V^N} t_i^C(v) \geqslant -l_i$ for all $i \in N$. Notice that

$$t_i^C(v) = \begin{cases} -\frac{n-1}{n}\left[v_i - \psi(v_i)\right] - \phi_i & \text{if } \rho(v) = i \\ \frac{1}{n}\left[v_j - \psi(v_j)\right] - \phi_i & \text{if } \rho(v) = j \neq i. \end{cases}$$

The following lemma is useful to determine the minimum of t_i^C.

Lemma 1 *For all* $k \in N$, $[v_k - \psi(v_k)]$ *is nonnegative and increasing in* $v_k \in V$.

Proof. See Appendix A. ■

From Lemma 1 it is clear that the minimum of t_i^C is attained when $\rho(v) = i$ and $v_i = \bar{v}$. Therefore,

$$\begin{aligned}\min_{v\in V^n} t_i^C(v) &= -\frac{n-1}{n}\left[\bar{v} - \psi(\bar{v})\right] - \phi_i \\ &= -\frac{n-1}{n}G - \phi_i. \end{aligned} \tag{11}$$

EPCC is then equivalent to $-\frac{n-1}{n}G - \phi_i \geqslant -l_i$ for all $i \in N$.

Combining IIR and EPCC yields the following condition

$$\phi_i \leqslant \min\{C(r_i), l_i\} - \frac{n-1}{n}G.$$

Then, for each $i \in N$, let the constant be

$$\phi_i = \min\{C(r_i), l_i\} - \frac{1}{n}\sum_{j\in N}\min\{C(r_j), l_j\}.$$

It is straightforward that $\sum_{i\in N}\phi_i = 0$ so that EPBB holds for t_i^C. Furthermore, if condition (8) holds, *i.e.* $\sum_{j\in N}\min\{C(r_j), l_j\} \geqslant (n-1)G$, it is immediate that $\phi_i \leqslant \min\{C(r_i), l_i\} - \frac{n-1}{n}G$ so that IIR and EPCC also hold for t_i^C. ■

Remark. In the absence of cash constraints, *i.e.* when the l_i's are sufficiently large for each $i \in N$ so that $\min\{C(r_i), l_i\} = C(r_i)$ for all $i \in N$ and $r \in \Delta^{n-1}$, condition (8) simply rewrites

$$\sum_{i\in N} C(r_i) \geqslant (n-1)G.$$

It can easily be shown that this condition is simply equation (D) in CGK (Theorem 1, p. 619).

On The Transfer Function. The major difficulty in establishing Theorem 1 lies in the construction of the transfer function t_i^C to prove the sufficiency of condition (8). Unfortunately, the transfer function proposed by CGK – that I will denote t_i^{CGK} thereafter – cannot be used to prove the sufficiency part of Theorem 1. Investigating t_i^{CGK} allows for a better understanding of the construction of cash-constrained mechanisms. Furthermore, the transfer function t_i^{CGK} is not peculiar to the work of CGK and it is commonly used in the mechanism design literature.[14] Formally, t_i^{CGK} writes

$$t_i^{CGK}(v) = c_i - \int_{\underline{v}}^{v_i} x dF(x)^{n-1} + \frac{1}{n-1}\sum_{j\neq i}\int_{\underline{v}}^{v_j} x dF(x)^{n-1},$$

where $c_i \in \mathbb{R}$ is a constant and $\sum_{i\in N} c_i = 0$ (see CGK, p. 628). Imposing EPCC requires that $t_i^{CGK}(v) \geqslant -l_i$ for all $i \in N$ and $v \in V^n$. It is immediate that $\min_{v\in V} t_i^{CGK}(v) = c_i - \int_{\underline{v}}^{\overline{v}} x dF(x)^{n-1}$, that is when $v_i = \overline{v}$ and $v_j = \underline{v}$ for all $j \neq i$. Straightforward computations yield that t_i^{CGK} is EPCC if

$$c_i - \frac{n-1}{n}G - \int_{\underline{v}}^{\overline{v}} x[1-F(x)]dF(x)^{n-1} \geqslant -l_i. \tag{12}$$

Applying a similar reasoning to that of the proof of Theorem 1, it is easy to show that a mechanism (s^*, t_i^{CGK}) is EF, IIC, IIR, EPBB and EPCC if and only if

$$\sum_{i\in N}\min\left\{C(r_i), l_i - \int_{\underline{v}}^{\overline{v}} x[1-F(x)]dF(x)^{n-1}\right\} \geqslant (n-1)G. \tag{13}$$

[14]To the best of my knowledge, this function has first been introduced by D'Aspremont and Gérard-Varet (1979, Theorem 6, p. 38). It can also be found in Robert and Lu (2001), Ledyard and Palfrey (2007) and Segal and Whinston (2011) among others.

This condition is undoubtedly more restrictive than condition (8).[15] This is due to the fact that the minimum of the variable part of t_i^{CGK} (*i.e.* ignoring the constant c_i) is lower than the minimum of the variable part of t_i^C (*i.e.* ignoring the constant ϕ_i). It is therefore interesting to notice that what matters for EPCC is the minimum of the *variable part* of the transfer function. If the variable part of the transfer function is too low, adding a higher constant to shift the function up to satisfy EPCC does not help as this would contradict EPBB.

Therefore, it is necessary that t_i^C has a lower span than t_i^{CGK} to satisfy EPCC. Notice that IIC requires that both transfer functions must be equal at the interim level (up to a constant), that is $\mathbb{E}_{-i}t_i^{CGK}(v) = \mathbb{E}_{-i}t_i^C(v) + \text{cst}$ for some constant.[16] The problem of constructing t_i^C is then a problem of constructing multivariate random variables with given marginals (IIC), bounded support (EPCC) and such that the sum of all is zero (EPBB). In this paper, I have constructed the function t_i^C by adapting the one proposed in an unpublished paper of Dudek, Kim and Ledyard (1995) who study ex post individually rational Bayesian mechanisms with no initial endowments. In future works, it would be interesting to fully characterize the space of transfer functions that satisfy EPCC.

Dominant Strategy Mechanisms. As mentioned earlier, the same dissolution condition applies to dominant strategy mechanisms.

Theorem 2 *An EF, EPIC, IIR, EABB and EPCC dissolution mechanism exists **if and only if***

$$\sum_{i \in N} \min\{C(r_i), l_i\} \geqslant (n-1)G. \tag{14}$$

Proof. See Appendix A. ■

Condition (14) is exactly the same as condition (8). This implies that when it is possible to ex post efficiently dissolve a partnership with a Bayesian mechanism then it is also possible to do it with a dominant strategy, and reciprocally. The equivalence, however, goes further than that as discussed below.

3.3. *The Equivalence Theorem*

From Theorem 1 and Theorem 2 it is then clear that when it is possible to implement ex post efficient dissolution with a Bayesian mechanism then ex post efficient dissolution can also be implemented with a dominant strategy mechanism, and *vice versa*. Yet, the following results give a much stronger equivalence between the two classes of mechanisms.

Theorem 3 *If $(s^*, \tilde{t})$ is an EF, **EPIC**, IIR, **EABB** and EPCC dissolution mechanism, then there exists a t such that*

[15]See Section 5 for an example in which the dissolution mechanism proposed by CGK does not allow for efficient dissolution whereas it is possible with my mechanism.

[16]This is a consequence of Proposition 1 and one of the most important features of incentive compatibility. See Myerson (1981), Makowski and Mezzetti (1994) and Williams (1999).

1. (s^*, t) *is an EF,* ***IIC****, IIR,* ***EPBB*** *and EPCC dissolution mechanism;*
2. $\mathbb{E}_{-i}t_i(v_i, v_{-i}) = \mathbb{E}_{-i}\tilde{t}_i(v_i, v_{-i})$ *for all* $i \in N$, $v_i \in V$.

Proof. See Appendix A. ■

The converse is also true.

Theorem 4 *If* (s^*, t) *is an EF,* ***IIC****, IIR,* ***EPBB*** *and EPCC dissolution mechanism, then there exists a* $\tilde{t}$ *such that*

1. $(s^*, \tilde{t})$ *is an EF,* ***EPIC****, IIR,* ***EABB*** *and EPCC dissolution mechanism;*
2. $\mathbb{E}_{-i}t_i(v_i, v_{-i}) = \mathbb{E}_{-i}\tilde{t}_i(v_i, v_{-i})$ *for all* $i \in N$, $v_i \in V$.

Proof. See Appendix A. ■

The additional feature of the equivalence theorem relies on the equivalence between the interim transfers. Thus, Theorem 3 and 4 state that one can alternatively choose to implement ex post efficient dissolution with Bayesian or dominant strategy mechanisms *and* offer the same interim transfers and utilities to every agent. It means that any final distribution of welfare among agents that is attainable in Bayesian mechanisms is also attainable in dominant strategy mechanisms.

3.4. *Interim Cash Constraints*

So far, I have assumed the strictest requirement for cash constraints, namely, ex post cash constraints. An important question is whether relaxing the requirement from the ex post to the interim level helps relaxing the dissolution condition (8). Formally, the constraints would become $\mathbb{E}_{-i}t_i(v) \equiv T_i(v_i) \geqslant -l_i$ for all $v_i \in V$. This constraint therefore requires that each agent, when privately informed about their type, thinks that they will have enough cash in expectations. This is a softer budget constraint than ex post cash constraints as it may occur that agents have to pay more than l_i at the end of the game. The following result provides an important insight about interim cash constraints.

Proposition 2 *An EF, IIC, IIR, EPBB and EPCC exists* (s, t) ***if and only if*** *there exists an EF, IIC, IIR, EPBB and* ***interim cash-constrained*** *mechanism* $(s, \tilde{t})$ *such that* $\mathbb{E}_{-i}t_i(v_i, v_{-i}) = \mathbb{E}_{-i}\tilde{t}_i(v_i, v_{-i})$ *for all* $i \in N$ *and* $v_i \in V$.

Proof. See Appendix A.

Proposition 2 shows that the existence of dissolution mechanisms with interim cash constraints is equivalent to the existence of dissolution mechanisms with ex post cash constraints. Therefore, relaxing cash constraints to the interim level has no advantages compared to the ex post level. This result straightforwardly applies to dominant strategy mechanisms.

4. FIRST-BEST CHARACTERIZATION RESULTS

The condition that must hold to implement ex post efficient dissolution (equation (8)) depends both on the initial ownership structure r and on the cash resources l. It is then natural to investigate the set of partnerships that can be efficiently dissolved when r and l vary.

4.1. *Optimal Initial Ownership Structures*

For a given distribution of cash resources $l \in \mathbb{R}_+^n$, I first characterize the initial ownership structures that maximize the contributions that can be collected on agents $\sum_{i \in N} \min\{C(r_i), l_i\}$. Initial ownership structures $r^*(l) \in \arg\max_{r \in \Delta^{n-1}} \sum_{i \in N} \min\{C(r_i), l_i\}$ are said to be *optimal*. It may obviously be the case that even the optimal initial ownership structures do not allow for ex post efficient dissolution if cash resources are low for some agents.[17]

Recall that $C(r_i) = \inf_{v_i} \left\{\mathbb{E}_{-i} g(v) - v_i r_i\right\}$. Let $y = \max_{j \neq i} v_j$, then

$$\begin{aligned}\mathbb{E}_{-i} g(v) - v_i r_i &= v_i \mathbb{E}_{-i} \mathbb{1}\{v_i > y\} + \mathbb{E}_{-i} y \mathbb{1}\{v_i < y\} - v_i r_i \\ &= v_i F(v_i)^{n-1} + \int_{v_i}^{\overline{v}} y \, dF(y)^{n-1} - v_i r_i.\end{aligned}$$

Differentiating this expression with respect to v_i, the first-order condition gives $F(v_i^*(r_i))^{n-1} = r_i$, where $v_i^*(r_i)$ is said to be the *worst-off type* of agent i.[18] Therefore,

$$C(r_i) = \int_{v_i^*(r_i)}^{\overline{v}} y \, dF(y)^{n-1}, \tag{15}$$

which is continuous and differentiable in r_i. The Envelope Theorem directly gives $C'(r_i) = -v_i^*(r_i) \leqslant 0$ and $C''(r_i) = -\frac{\partial v_i^*}{\partial r_i}(r_i) < 0$ so that $C(r_i)$ is both decreasing and concave in r_i. Notice also that $C(0) = \int_{\underline{v}}^{\overline{v}} y \, dF(y)^{n-1}$ and $C(1) = 0$.

It is useful to introduce the following notation. Let $\tilde{r}_i \in [0, 1]$ be such that $C(\tilde{r}_i) = l_i$ when $l_i \leqslant C(0)$ and let $\tilde{r}_i = 0$ if $l_i > C(0)$. This threshold is such that cash constraints are more restrictive than the individual rationality constraint when $r_i < \tilde{r}_i$ and the opposite when $r_i \geqslant \tilde{r}_i$. As $\tilde{r}_i$ is decreasing in l_i, a higher $\tilde{r}_i$ indicates that cash constraints are more restrictive for agent i.

The characterization results depend both on the total amount of available cash resources and on its distribution over agents. Assume, without loss of generality, that $l_1 \geqslant \cdots \geqslant l_n$ so that $\tilde{r}_1 \leqslant \cdots \leqslant \tilde{r}_n$. Consider first the case in which cash constraints are not too severe, that is when $\sum_{i \in N} \tilde{r}_i \leqslant 1$.

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[17]This happens when $\sum_{i \in N} \min\{C(r_i^*(l)), l_i\} < (n-1)G$. It is then possible to characterize the minimal subsidy that would be necessary to satisfy the ex post efficient dissolution condition (8).

[18]The second-order derivative immediately writes $(n-1) f(v_i) F(v_i)^{n-2} \geqslant 0$ so that the first-order condition characterizes a minimum.

Proposition 3 *Assume* $\sum_{i\in N}\tilde{r}_i \leqslant 1$*, then the optimal distribution of property rights* $r^* \in \Delta^{n-1}$ *is as follows:*

a. If $\tilde{r}_i \leqslant \frac{1}{n}$ *for all* $i \in N$ *, then* $r^* = (\frac{1}{n}, \ldots, \frac{1}{n})$*;*

b. If $\tilde{r}_i > \frac{1}{n}$ *for some* $i \in N$*, then* $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r},\ \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ *where* $\hat{r} = \frac{1-\sum_{j\geqslant p}\tilde{r}_j}{p-1}$ *for some* $p \in N$ *such that* $\max_{i<p}\tilde{r}_i < \hat{r} \leqslant \min_{j\geqslant p}\tilde{r}_j$*.*

Proof. See Appendix A. ■

Proposition 3.a is simply CGK main result (Proposition 1, p.621). When each agent is endowed with enough cash, *i.e.* $\tilde{r}_i \leqslant \frac{1}{n}$, then the equal-share ownership structure is optimal. However, as soon as at least one agent's cash resources go below some threshold, *i.e.* $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$, Proposition 3.b implies that the optimal ownership structure allocates more initial ownership rights to more cash-constrained agents.

To illustrate Proposition 3.b , consider a two-agent partnership in which agent 1 has large cash resources so that $\tilde{r}_1 = 0$ and agent 2 is heavily cash constrained so that $\tilde{r}_2 \in [\frac{1}{2}, 1)$. It is clear that starting from any $r_2 < \tilde{r}_2$, and in particular $r_2 = \frac{1}{2}$, it would be possible to strictly increase $\sum_{i=1,2}\min\{C(r_i), l_i\} = C(r_1) + l_2$ by increasing r_2 up to $\tilde{r}_2$ as $C(\cdot)$ is a decreasing function and $\min\{C(r_2), l_2\} = l_2$ in unchanged for all $r_2 \leqslant \tilde{r}_2$. In other words, it is innocuous to give more initial ownership rights to heavily cash-constrained agent as they are already limited by their cash resources but it allows to give less initial ownership rights to less cash-constrained agents and then collect more from them.

When cash constraints are more severe, for some or all agents, so that $\sum_{i\in N}\tilde{r}_i > 1$, the structure of the optimal initial ownership structure can be characterized as follows.

Proposition 4 *Assume* $\sum_{i\in N}\tilde{r}_i \geqslant 1$*, then the optimal distribution of property rights* $r^* \in \Delta^{n-1}$ *is such that* $r_i^* \leqslant \tilde{r}_i$ *for all* $i \in N$ *and* $\sum_{i\in N}\min\{C(r_i^*), l_i\} = \sum_{i\in N}\min\{C(0), l_i\}$*.*

Proof. First, notice the following upper bound, $\sum_{i\in N}\min\{C(r_i), l_i\} \leqslant \sum_{i\in N}\min\{C(0), l_i\}$ for all $r \in \Delta^{n-1}$. For every $i \in N$, let $r_i \leqslant \tilde{r}_i$ which is always possible as $\sum_{i\in N} r_i = 1 \leqslant \sum_{i\in N}\tilde{r}_i$. Then $\min\{C(r_i), l_i\} = \min\{C(0), l_i\}$ for all $i \in N$ and $\sum_{i\in N}\min\{C(r_i), l_i\} = \sum_{i\in N}\min\{C(0), l_i\}$. To conclude, it is clear that choosing any $r_i > \tilde{r}_i$ would decrease $\sum_{i\in N}\min\{C(r_i), l_i\}$. ■

When $\sum_{i\in N}\tilde{r}_i \geqslant 1$, optimal initial ownership structures may or may not have a clear characterization. Consider for instance the case in which all agents $i < p$ for some $p \in N \setminus \{1\}$ have large cash resources so that $\tilde{r}_i = 0$ for all $i < p$ and all agents $i \geqslant p$ have low cash resources so that $\tilde{r}_i > 0$ and $\sum_{i\in N}\tilde{r}_i \geqslant 1$. Then, Proposition 4 gives that all agents $i < p$ must have $r_i^* = 0$ and for $i \geqslant p$, the r_i^* must be such that $\sum_{i\geqslant p} r_i^* = 1$. Therefore, initial ownership goes only to agents with few cash resources and this resembles Proposition 3.b. Consider now a case in which for instance $\frac{1}{2} = \tilde{r}_1 = \tilde{r}_2 \leqslant \tilde{r}_3 \leqslant \cdots \leqslant \tilde{r}_n$. Then choosing $r_1^* = r_2^* = \frac{1}{2}$ and $r_i^* = 0$ for all $i \geqslant 3$ is optimal. Hence, at some point, cash constraints are so severe that optimal initial ownership structures have no clear structure other than that of Proposition 4.

4.2. *Minimal Cash Resources To Dissolve Equal-Share Partnerships*

One of the main results of CGK states that equal-share partnership can always be ex post efficiently dissolved. De Frutos and Kittsteiner (2008) report that one-half to two-third of two-agent partnerships exhibit equal-share ownership. It is then interesting to investigate how cash constraints mitigate this finding.

Proposition 5 *Assume $l_i := \underline{l}$ for all $i \in N$. Then, every equal-share partnership is dissolvable for any absolutely continuous cumulative distribution function $F(\cdot)$ if and only if*

$$\underline{l} \geqslant \frac{n-1}{n} G. \tag{16}$$

Furthermore, for any $F(\cdot)$, $\underline{l}$ is increasing in n and converges to $\overline{v}$ when n goes to infinity.

Proof. From CGK (Proposition 1), an equal-share partnership is always dissolvable in the absence of cash constraints, that is, $\sum_{i \in N} C(\frac{1}{n}) \geqslant (n-1)G$. Equation (16) immediately follows from the dissolution condition $\sum_{i \in N} \min\{C(\frac{1}{n}), \underline{l}\} \geqslant (n-1)G$.

Recall that $G = \mathbb{E}[\max_{j \in N} v_j] = \int_{\underline{v}}^{\overline{v}} y dF(y)^n$. Differentiating the right-hand side of (16) with respect to n gives $\frac{1}{n^2}G + \frac{n-1}{n}\frac{\partial G}{\partial n} \geqslant 0$ as G is increasing in n. Finally, if $n \to \infty$ then $G \to \overline{v}$ and so does $\frac{n-1}{n}G$. ■

Equation (16) simply states that in equal-share-equal-cash partnerships, each agents must be endowed with a fraction $\frac{1}{n}$ of the total ex ante expected deficit generated by a Groves mechanism. By construction, this ex ante deficit becomes larger as the number of agents increases as the probability of $\max_{i \in N} v_i$ increases (which must be distributed to $(n-1)$ agents, see Section 3) and so do the minimal cash resources.

Finally, the result that $\underline{l} \to \overline{v}$ when $n \to +\infty$ suggests that cash constraints are likely to be a major concern as the number of agents becomes large. In many cases, it seems reasonable to think that one of the main purposes of forming a partnership is precisely to split the burden of a costly investment between partners because of initial cash constraints. Hence, it seems unlikely that every agent possesses the maximal value of the asset in cash when dissolution occurs.

4.3. *Some examples*

To illustrate Propositions 3, 4 and 5, assume that the asset value is between 0 and 1 million and valuations are uniformly distributed over support $[0, 1]$.

EXAMPLE 1. Let $n = 3$ and assume $\tilde{r} = (0, 0.4, 0.5)$. This happens when cash resources are approximately $(l_1, l_2, l_3) \approx (0.66, 0.49, 0.43)$. Assuming equal-share ownership gives $\sum_{i=1,2,3} \min\{C(\frac{1}{3}), l_i\} = C(\frac{1}{3}) + l_2 + l_3 \approx 1.47$ as $C(\frac{1}{3}) \approx 0.54 > l_2 > l_3$. Given that $(n-1)G = \frac{3}{2}$, it follows that the equal-share ownership structure does not allow for ex post efficient dissolution.

Instead, as $\sum_{i\in N}\tilde{r}_i = 0.9$ and $\frac{1}{3} < \tilde{r}_2 < \tilde{r}_3$, Proposition 3.b gives that the optimal ownership structure writes $\tilde{r}^* = (0.1, 0.4, 0.5)$. As $\sum_{i=1,2,3}\min\{C(\frac{1}{3}), l_i\} \approx 1.57 > \frac{3}{2}$, the optimal ownership structures allows for ex post efficient dissolution. Ownership rights are then inversely proportional to cash resources and the optimal ownership structure is quite asymmetric.

Example 2. Let $n = 3$ and assume $\tilde{r} = (0.3, 0.4, 0.45)$. This corresponds to $(l_1, l_2, l_3) \approx (0.56, 0.50, 0.47)$. Notice that $\sum_{i=1,2,3}\tilde{r}_i = 1.15 > 1$ so that Proposition 4 applies and thus $r_i^* \leqslant \tilde{r}_i$ for $i = 1, 2, 3$. As $C(0) = \frac{2}{3} > l_1 > l_2 > l_3$, it follows that $\sum_{i=1,2,3}\min\{C(r_i), l_i\} = l_1 + l_2 + l_3 \approx 1.52 > \frac{3}{2}$ and the partnership can be ex post efficiently dissolved. Notice that choosing for instance $r^* = (0.3, 0.3, 0.4)$ or $r^* = (0.3, 0.25, 0.45)$ both give $r_i^* \leqslant \tilde{r}_i$ for all $i \in N$ and achieve the same outcome. However, agent 2 receives less initial ownership rights than agent 1 whereas the former has larger cash resources than the latter. This illustrates that optimal ownership structures might not always exhibit an inversely proportional relationship between cash and ownership rights when cash resources are severe, *i.e.* $\sum_{i\in N}\tilde{r}_i \geqslant 1$.

Example 3. Assume equal-share ownership, $r_i = \frac{1}{n}$ for all $i \in N$ and symmetric cash resources $l_i = \tilde{l}$ for all $i \in N$. According to Proposition 5, ex post efficient dissolution is possible if and only if $\tilde{l} \geqslant \frac{n-1}{n}G = \frac{n-1}{n+1}$. Hence, when $n = 2, 3, 4, 5$ and 6, the minimal cash resources are respectively $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$ and $\frac{5}{7}$. A partnership with 5 agents for instance, each partner must possess two-third of a million to achieve ex post efficient dissolution.

5. IMPLEMENTATION: A SIMPLE AUCTION

The ex post efficient dissolution mechanism presented in Section 3 is appealing for its convenient mathematical properties which greatly simplifies the analysis of such mechanisms. It is, however, less appealing for the practitioner as it requires the setup of a game in which each agent reports their valuation and is communicated the transfer function $t_i^C(v)$.

Ex Post Efficient Bidding Game. I propose a simple auction that replicates the ex post efficient dissolution mechanism whenever the ex post efficient dissolution condition (8) holds. Each agent proposes a bid and the highest bidder receives full ownership of the asset. The auction is designed such that the bidding strategies are increasing in the agents' valuations so that the highest bidder coincides with the agent with the highest valuation. Each agent also receives side payments conditional on their share of ownership rights and cash resources to ensure IIR, EPBB and EPCC.

Let $b := (b_1, \dots, b_n) \in \mathbb{R}_+^n$ denote the vector of bids. In the ex post efficient bidding game each agent receives a side payment $\phi_i(r, l)$ and pays a price $p_i(b_1, \dots, b_n)$.

Theorem 5 *A bidding game with prices*

$$p_i(b_1, \dots, b_n) := \begin{cases} (n-1)\left[b_i + \frac{1}{n}\underline{v}\right] & \text{if } b_i \geqslant \max_k b_k \\ -\left[b_j + \frac{1}{n}\underline{v}\right] & \text{if } b_j \geqslant \max_k b_k, \end{cases}$$

and side-payments

$$\phi_i(r, l) := \frac{1}{n}\sum_{j \in N} \min\{C(r_j), l_j\} - \min\{C(r_i), l_i\},$$

efficiently dissolves any dissolvable partnership with cash-constrained agents.

Proof. Let b_i be player i's strategy and $b(v_j)$ be the bidding strategy of player $j \neq i$ with valuation v_j. Agent i's interim expected utility (omitting side payments) when bidding b_i writes

$$U_i(b_i; v_i) := \left[v_i - (n-1)(b_i + \frac{1}{n}\underline{v})\right] \mathbb{E}_{-i}\mathbb{1}\{b_i > \max_{k \neq i} b(v_k)\}$$
$$+ \sum_{j \neq i} \mathbb{E}_{-i}\left[\mathbb{1}\{b(v_j) > b_i\}\mathbb{1}\{b(v_j) > \max_{k \neq i,j} b(v_k)\}\left[b(v_j) + \frac{1}{n}\underline{v}\right]\right].$$

Solving for strictly increasing symmetric Bayesian equilibrium, the bidding strategy of the $j \neq i$ players, $b(v_j)$, is strictly increasing and then invertible. Notice that $\mathbb{1}_{\{b_i > \max_{k \neq i} b(v_k)\}} = \mathbb{1}_{\{b^{-1}(b_i) > \max_{k \neq i} v_k\}}$ and $\mathbb{1}_{\{b(v_j) > \max_{k \neq i} b(v_k)\}} = \mathbb{1}_{\{v_j > \max_{k \neq i} v_k\}}$. It follows that player i's interim expected utility rewrites

$$U_i(bi; v_i) = \left[v_i - (n-1)(b_i + \frac{1}{n}\underline{v})\right] Z(b^{-1}(b_i)) + \int_{b^{-1}(b_i)}^{\overline{v}} \left[b(v_j) + \frac{1}{n}\underline{v}\right] dZ(v_j),$$

where $Z := F^{n-1}$. Let $z = Z'$, differentiating $U(bi; v_i)$ with respect to b_i and simplifying using $\frac{\partial b^{-1}}{\partial b_i}(b_i) = \frac{1}{b'(b^{-1}(b_i))}$ gives

$$\frac{\partial U_i}{\partial b_i}(b_i; v_i) = -(n-1)Z(b^{-1}(b_i)) + \frac{z(b^{-1}(b_i))}{b'(b{-}1(b_i))}\left[v_i - nb_i - \underline{v}\right].$$

At equilibrium, $b(v_i)$ must be such that $\frac{\partial U_i}{\partial b_i}(b(v_i); v_i) = 0$. Therefore, $b(v_i)$ must solve

$$-(n-1)Z(v_i) + \frac{z(v_i)}{b'(v_i)}\left[v_i - nb(v_i) - \underline{v}\right] = 0.$$

It is easy to show that $b(v_i) := \int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt$ solves this first-order differential equation and is strictly increasing in v_i.[19] It follows that at the Bayesian equilibrium, player i pays a

[19] The first-order condition is also sufficient. Notice that $b(v_i) = \int_{\underline{v}}^{v_i} \psi(t)\frac{f(t)}{F(t)}dt$ where $\psi(t)$ is defined by equation (10). Then $b'(v_i) = \psi(v_i)\frac{f(v_i)}{F(v_i)} = (v_i - nb(v_i) - \underline{v})\frac{f(v_i)}{F(v_i)}$, where the second equality stems from Lemma 1. Hence,

price

$$p_i(b(v_1),\dots,b(v_n)) = \begin{cases} (n-1)\left[\int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_i \geqslant \max_k b_k \\ -\left[\int_{\underline{v}}^{v_j} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_j \geqslant \max_k b_k. \end{cases} \quad (17)$$

It can easily be proven that $p_i(b(v_1),\dots,b(v_n))$ together with $\phi_i(r,l)$ exactly replicates the transfer rule of Theorem 1, $t_i^C(v)$, for all $v \in V^n$ and all $i \in N$. The bidding game is thus EF (as $b(\cdot)$ is increasing, the bidder with the highest valuation gets full ownership), IIR, EPBB and EPCC as it reproduces the transfer rule of Theorem 1. ■

Theorem 5 can be interpreted as follows. The agent with the highest bid, $b_k = \max_{i\in N} b_i$, receives full ownership of the asset and gives an amount of money equal to b_k to all other agents $j \in N \setminus \{k\}$. Unconditional on bids, each agent receives a transfer $\phi_i(r,l)$.

Furthermore, there is no need for an outside party to advance money before running the auction. In practice, the auctioneer could simply announce to each agent their side payment $\phi_i(r,l)$, run the auction and then compute total payment for each agent, $\phi_i(r,l) - p_i(b_1,\dots,b_n)$ from the submitted bids. Agents with a $\phi_i(r,l) - p_i(b_1,\dots,b_n) < 0$ pay the auctioneer who then redistribute this amount of money to the agents with $\phi_i(r,l) - p_i(b_1,\dots,b_n) \geqslant 0$. As the auction is budget balanced, this can always be done without outside funding.

Comparison With CGK: Uniform and Symmetric Example. It is interesting to investigate the properties of the bidding game proposed in Theorem 5 with the one proposed by CGK (Theorem 2, p. 620). In CGK, the bidding game has prices $p_i^{CGK}(b) = b_i - \frac{1}{n-1}\sum_{j\neq i} b_j$ and side payments $c_i^{CGK}(r) = \int_{\underline{v}}^{v_i^*} x dF(x)^{n-1} - \frac{1}{n}\sum_{j\in N}\int_{\underline{v}}^{v_j^*} x dF(x)^{n-1}$ where v_i^* is defined as in Section 4.1. The equilibrium bidding strategy in CGK is given by $b^{CGK}(v_i) = \int_{\underline{v}}^{v_i} u dF(u)^{n-1}$.

For simplicity, assume that valuations are uniformly distributed over support $[0,1]$, and consider an equal-share-equal-cash partnership, *i.e.* $r_i = \frac{1}{n}$ and $l_i = \tilde{l}$ for all $i \in N$. It immediately follows that $\phi_i(r,l) = c_i^{CGK}(r) = 0$ for all $i \in N$, that is, side payments are zero for all agents in both bidding games. In the cash-constrained auction, the maximal price is obtained by maximizing equation (17), which is the same as minimizing t_i^C (ignoring the constant term) as the auction replicates the dissolution mechanism. Therefore, from equation (11), $\max_{v\in V^n} p_i(b) = \frac{n-1}{n}G = \frac{n-1}{n+1}$ as $G = \frac{n}{n+1}$ in the uniform case. Alternatively, using the LHS of equation (12) (again ignoring the constant) gives that $\max_{v\in V^n} p_i^{CGK}(b) = \frac{n-1}{n}$. Hence, it is clear that $\max_{v\in V^n} p_i^{CGK}(b) > \max_{v\in V^n} p_i(b)$ so that the maximal price in CGK auction is strictly higher than the one in the cash-constrained auction.

after simplifications, $\frac{\partial U_i}{\partial b_i}(b_i;v_i) = (n-1)Z(b^{-1}(b_i))\left[-1 + \frac{v_i - nb_i - \underline{v}}{b^{-1}(b_i) - nb_i - \overline{v}}\right]$. It follows that $\frac{\partial U_i}{\partial b_i}(b_i;v_i) \geqslant 0$ (resp. $\leqslant 0$) when $b_i \leqslant b(v_i)$ (resp. $\geqslant b(v_i)$) for any $v_i \in V$.

Given that side payments are null in both auctions, EPCC requires that the maximal price never exceeds the agents' financial resources $\tilde{l}$. Let $n = 2$, then $\max_{v \in V^n} p_i^{CGK}(b) = \frac{1}{2}$ and $\max_{v \in V^n} p_i^{CGK}(b) = \frac{1}{3}$. It follows that any of the two auctions ex post efficiently dissolves this symmetric partnership for $\tilde{l} \geqslant \frac{1}{2}$ and none dissolves it if $\tilde{l} < \frac{1}{3}$ (this indeed violates condition (8)). However, only the cash-constrained auction ex post efficiently dissolves this partnership for $\tilde{l} \in \left[\frac{1}{3}, \frac{1}{2}\right)$ as CGK auction requires agents to pay prices that may exceed their financial resources. This particular example illustrates the discrepancy between the necessary and sufficient condition (8) and the sufficient dissolution condition with CGK mechanism, condition (13).

6. SECOND-BEST DISSOLUTION MECHANISMS

When condition (8) is not satisfied, it is not possible to simultaneously ensure an ex post efficient allocation rule and all the required constraints. Therefore, the problem consists in finding a new mechanism whose allocation rule simultaneously maximizes the ex ante surplus and satisfies all the constraints. These mechanisms are referred to as *second-best mechanisms*.

One of the most difficult challenges when investigating second-best mechanisms comes from the participation and cash constraints. Usually, the structure of the problems studied in the literature are such that those types of constraints can be replaced by the one for the lowest/highest type independently of the mechanism in question. In partnership models, however, the worst-off types are endogenously determined by the mechanism and must be defined simultaneously to the allocation rule.

Second-best mechanisms in partnership models have mainly been studied by Lu and Robert (2001) and Loertscher and Wasser (2019) who consider Bayesian mechanisms without cash constraints. I build on these two papers and on Boulatov and Severinov (2018) who investigate the design of optimal auctions in the presence of cash-constrained bidders.

METHODOLOGY AND ASSUMPTIONS. I investigate dissolution mechanisms satisfying IIC, IIR, EPCC but I relax EPBB to EABB as imposing EPBB had proved too difficult. I conjecture, however, that adding the EPBB requirement can directly be derived using the present work and only requires to construct an appropriate transfer function. I hope that future work will address this issue.

The first-best analysis shows that implementing the ex post efficient allocation rule requires that enough money can be collected on agents to cover the cost of imposing incentive compatibility. The same logic applies to any other allocation rule. Following the methodology of Lu and Robert (2001) and Loertscher and Wasser (2019), I start by investigating allocation rules that maximize a linear combination of the expected surplus and the expected revenue that can be collected on agents by imposing IIC, IIR and EPCC but ignoring EABB. This allows to characterize optimal second-best mechanisms for any possible budget deficit.[20] It then appears that solving the second-best problem including EABB is just a particular solution to the previous problem for well-chosen weights on expected surplus and collected revenue.

Then, for any weight $\lambda \in [0,1]$, the objective function writes,

$$W_\lambda := (1-\lambda) \sum_{i \in N} \mathbb{E}\big[v_i(s_i(v) - r_i)\big] + \lambda \sum_{i \in N} \mathbb{E}\big[-t_i(v)\big], \tag{18}$$

where the first term is the expected surplus and the second term is the expected revenue. I introduce the following two notations. For any $\lambda \in [0,1]$, let

$$\alpha(v_i \mid \lambda) = v_i - \lambda \frac{1-F(v_i)}{f(v_i)} \quad \text{and} \quad \beta(v_i \mid \lambda) = v_i + \lambda \frac{F(v_i)}{f(v_i)},$$

where $\alpha(\cdot \mid \lambda)$ and $\beta(\cdot \mid \lambda)$ are referred to as *buyer's virtual valuation* and *seller's virutal valuation*, respectively (see Lu and Robert (2001)). Notice that for any $v_i \in (\underline{v}, \overline{v})$ and $\lambda \in [0,1]$, $\alpha(v_i \mid \lambda) < v_i < \beta(v_i \mid \lambda)$. The following assumptions are made on these functions.

Assumption 1 *For any $\lambda \in [0,1]$, the virtual valuations $\alpha(v_i \mid \lambda)$ and $\beta(v_i \mid \lambda)$ are both strictly increasing in v_i.*

This assumption is a regularity assumption on the distribution function $F(\cdot)$ to avoid bunching due to nonregular distribution functions.[21] It is standard in the literature (see Myerson, 1981) and weaker than imposing increasing hazard rate.

Assumption 2 *Assume that $f(\cdot)$ is nonincreasing.*

This assumption is not necessary as suggested by Boulatov and Severinov (2018) but it is imposed in the present work as it greatly simplifies the analysis.

6.1. *Characterization Of General IIC, IIR And EPCC Mechanisms*

The analysis of dissolution mechanisms is no longer restricted to the ex post efficient allocation rule s^* defined in Section 3. Therefore, Proposition 1 does not apply here and it is necessary to characterize dissolution mechanisms satisfying IIC, IIR and EPCC for any possible allocation rule.

Incentive Compatibility. Take any allocation rule $s(v) = (s_1(v), \dots, s_n(v))$ such that $s_i(v) \in [0,1]$ for all $i \in N$, $v \in V^n$ and $\sum_{i \in N} s_i(v) = 1$ for all $v \in V^n$. The following standard characterization of IIC mechanisms applies (see Myerson (1981)).

[20]It may sometimes not be desirable to strictly impose budget balance. For instance, if the dissolution problem is run by a public authority, it may be willing to achieve a more efficient outcome at the expense of covering a strictly positive deficit.

[21]However, second-best mechanisms will generically involve bunching due to the initial allocation of ownership rights and to the cash constraints. This assumption on the distribution of valuation therefore only rules out bunching due to nonregular distribution functions.

Lemma 2 *A dissolution mechanism* (s, t) *is IIC if and only if*

$$S_i \text{ is nondecreasing for all } i \in N, \tag{IC1}$$

$$U_i(v_i) = U_i(v_i^*) + \int_{v_i^*}^{v_i} (S_i(x) - r_i) dx \quad \text{for all } v_i, v_i^* \in V. \tag{IC2}$$

Proof. The proof is standard and thus omitted (see Myerson, 1981; Lu and Robert, 2001). ■

The set of dissolution mechanisms satisfying IIC must then be such that the interim allocation rule $S_i := \mathbb{E}_{-i} s_i$ is nondecreasing and the interim utility of each agent must satisfy (IC2). Lemma 2 also implies that U_i is absolutely continuous. Recall that, by definition, $U_i(v_i) = v_i(S_i(v_i) - r_i) + T_i(v_i)$ so that (IC2) indirectly defines the interim transfers as follows

$$T_i(v_i) = T_i(v_i^*) - \int_{v_i^*}^{v_i} x dS_i(x) \quad \text{for all } v_i, v_i^* \in V, \tag{19}$$

which is decreasing in v_i.

Individual Rationality. Imposing IIR requires that $U_i(v_i) \geqslant 0$ for all $i \in N$ and $v_i \in V$ where $U_i(v_i)$ is defined by (IC2). Let $v_i^* \in \arg\min_{\tilde{v}_i \in V} U_i(\tilde{v}_i)$ denote a *worst-off* type for agent i, then IIR can be characterized as follows (see CGK and Lu and Robert (2001)).

Lemma 3 *A dissolution mechanism* (s, t) *is IIC and IIR if and only if it satisfies Lemma 2 and for every* $i \in N$,

$$U_i(v_i^*) \geqslant 0, \tag{IR1}$$

where v_i^* *denotes a worst-off type of agent* i *and*

$$v_i^* \in V^*(S_i) := \{v_i \mid S_i(x) \leqslant r_i, \forall x < v_i;\ S_i(y) \geqslant r_i, \forall y > v_i\}. \tag{IR2}$$

Proof. The proof is the same as in Lu and Robert (2001). ■

(IR1) simply states that the continuum of constraints $U_i(v_i) \geqslant 0$ can be replaced by imposing IIR only on the set of worst-off types $V^*(S_i)$ and (IR2) defines this set. Notice that (i) the set of worst-off types is endogenously determined by the interim allocation rule S_i and, (ii) this set contains all agents who are expected to be neither a buyer nor a seller.

Cash Constraints. Finally, imposing EPCC requires that $t_i(v) \geqslant -l_i$ for all $i \in N$ and $v \in V^n$. As first noted by Laffont and Robert (1996), it is possible to set a mechanism (s, t) where $t_i(v)$ depends only on agent i's private information so that $t_i(v) = T_i(v_i)$ for all $v \in V^n$. It affects neither the objective function W_λ nor IIC nor IIR as they all depend only on the interim transfers $T_i(v_i)$. Therefore, without loss of generality, EPCC is satisfied by requiring $T_i(v_i) \geqslant -l_i$ for all $i \in N$, $v_i \in V$.

Following Boulatov and Severinov (2018), define

$$\overline{m}_i := \inf\{v_i \in V \mid T_i(v_i) = T_i(\bar{v})\}. \tag{20}$$

Then, any dissolution mechanism (s,t) for which $\overline{m}_i < \bar{v}$ is such that $T_i(v_i) = T_i(\bar{v})$ for all $v_i \in [\overline{m}_i, \bar{v}]$ as IIR imposes that $T_i(v_i)$ is nondecreasing in v_i. As S_i is nondecreasing it follows from (19) that imposing $T_i(v_i) = T_i(\bar{v})$ for all $v_i \in [\overline{m}_i, \bar{v}]$ requires that $S_i(v_i)$ is constant over $[\overline{m}_i, \bar{v}]$. Then, without loss of generality, $S_i(v_i) = S_i(\overline{m}_i)$ for all $v_i \in [\overline{m}_i, \bar{v}]$ must hold for interim transfer to be constant.

EPCC can therefore be replaced with the following two conditions for all $i \in N$

$$T_i(\overline{m}_i) \geqslant -l_i, \tag{CC1}$$

$$S_i(v_i) = S_i(\overline{m}_i) \quad \text{for all } v_i \in [\overline{m}_i, \bar{v}]. \tag{CC2}$$

$(CC1)$ ensures that all transfers are lower or equal to each agent's cash resources while $(CC2)$ is necessary for interim transfers to stay constant when $v_i \in [\overline{m}_i, \bar{v}]$.

Notice that $(CC1)$ can also be expressed in terms of interim utility, that is, $T_i(\overline{m}_i) = U_i(\overline{m}_i) - \overline{m}_i\big(S_i(\overline{m}_i) - r_i\big) \geqslant -l_i$. Replacing $U_i(\overline{m}_i)$ by $(IC2)$ evaluated at $\overline{m}_i$ gives that $(CC1)$ rewrites

$$U_i(v_i^*) \geqslant \overline{m}_i\big(S_i(\overline{m}_i) - r_i\big) - \int_{v_i^*}^{\overline{m}_i} (S_i(x) - r_i)dx - l_i, \tag{CC1}$$

so that $(CC1)$ is expressed only in terms of the interim utility of a worst-off type.

6.2. *The Second-Best Optimization Program*

The problem therefore consists in maximizing W_λ subject to $(IC1)$, $(IC2)$, $(IR1)$, $(IR2)$, $(CC1)$, $(CC2)$ and the two resource constraints $s_i(v) \in [0,1]$ for all $v_i \in V$ and $\sum_{i\in N} s_i(v) = 0$ for all $v \in V^n$. First, notice that the transfer function only enters the objective function but can be completely removed from the problem by directly imposing $(IC2)$ on W_λ. Rewrite,

$$W_\lambda = \sum_{i\in N} \int_V \Big\{(1-\lambda)v_i(S_i(v_i) - r_i) - \lambda T_i(v_i)\Big\} dF(v_i).$$

$(IC2)$ gives that $T_i(v_i) = U_i(v_i^*) + \int_{v_i^*}^{v_i}(S_i(x) - r_i)dx - v_i[S_i(v_i) - r_i]$. Plugging this expression in the objective function, integrating by parts and rearranging yields

$$W_\lambda = \sum_{i\in N} \int_V (S_i(v_i) - r_i)\Big[v_i + \mathbb{1}\{v_i \leqslant v_i^*\}\lambda\frac{F(v_i)}{f(v_i)} - \mathbb{1}\{v_i \geqslant v_i^*\}\lambda\frac{1-F(v_i)}{f(v_i)}\Big] dF(v_i) - \lambda\sum_{i\in N} U_i(v_i^*).$$

Now I want to impose $(CC2)$ directly on the objective function. For that matter, first consider the following result.

Lemma 4 *For any IIC, IIR and EPCC mechanism with* $l \in (\mathbb{R}_+^*)^n$, *if* $\overline{m}_i < \overline{v}$ *then* $\overline{m}_i \geqslant \sup V^*(S_i)$.

Proof. As $\overline{m}_i < \overline{v}$ then $T_i(\overline{m}_i) = -l_i$ and then $U_i(\overline{m}_i) = \overline{m}_i(S_i(\overline{m}_i) - r_i) - l_i$. IIR implies that $U_i(\overline{m}_i) = \overline{m}_i(S_i(\overline{m}_i) - r_i) - l_i \geqslant 0$ which immediately requires that $S_i(\overline{m}_i) > r_i$. By definition of $V^*(S_i)$, $S_i(v_i) \leqslant r_i$ for all $v_i \leqslant \inf V^*(S_i)$ and $S_i(v_i) = r_i$ for all v_i in the interior of $V^*(S_i)$. It follows that $\overline{m}_i \geqslant \sup V^*(S_i)$. ■

As lemma 4 ensures that $\overline{m}_i \geqslant \sup V^*(S_i)$, plugging (CC2) into the objective function and using the definitions of $\beta(v_i \mid \lambda)$ and $\alpha(v_i \mid \lambda)$ yields

$$W_\lambda(s, U, \overline{m}) := \sum_{i \in N} \int_{\underline{v}}^{\overline{m}_i} (S_i(v_i) - r_i)\Big[\mathbb{1}\{v_i \leqslant v_i^*\}\beta(v_i \mid \lambda) + \mathbb{1}\{v_i \geqslant v_i^*\}\alpha(v_i \mid \lambda)\Big] dF(v_i) + \sum_{i \in N} \int_{\overline{m}_i}^{\overline{v}} (S_i(\overline{m}_i) - r_i)\alpha(v_i \mid \lambda) dF(v_i) - \lambda \sum_{i \in N} U_i(v_i^*),$$

where $W_\lambda(s, U, \overline{m})$ defines the objective as a function of the allocation rule s, the interim utilities of the worst-off types $U = (U_1(v_1^*), \ldots, U_n(v_n^*))$ and the thresholds $\overline{m} := (\overline{m}_1, \ldots, \overline{m}_n)$.

The Relaxed Problem. As it is standard in the literature, consider a relaxed problem in which (IC1) and (IR2) are ignored. It will be proven later that the relaxed problem satisfies those constraints. Let $\chi := (\chi_1, \ldots, \chi_n) \in \mathbb{R}_+^n$ and $\tau := (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ denote the Lagrange multipliers associated with (IR1) and (CC1), respectively. Then, the relaxed problem is given by the following Lagrangian

$$\mathcal{L} := W_\lambda(s, U, \overline{m}) + \sum_{i \in N} \chi_i U_i(v_i^*) + \sum_{i \in N} \tau_i \Big(U_i(v_i^*) - \overline{m}_i\big(S_i(\overline{m}_i) - r_i\big) + \int_{v_i^*}^{\overline{m}_i} (S_i(x) - r_i) dx + l_i\Big).$$

A little algebra shows that the Lagrangian can be rewritten

$$\mathcal{L} = \sum_{i \in N} \int_{\underline{v}}^{\overline{m}_i} (S_i(v_i) - r_i)\Big[\mathbb{1}\{v_i \leqslant v_i^*\}\beta(v_i \mid \lambda) + \mathbb{1}\{v_i \geqslant v_i^*\}\Big[\alpha(v_i \mid \lambda) + \frac{\tau_i}{f(v_i)}\Big] dF(v_i) + \sum_{i \in N} \int_{\overline{m}_i}^{\overline{v}} (S_i(\overline{m}_i) - r_i)\Big[\alpha(v_i \mid \lambda) - \frac{\tau_i \overline{m}_i}{1 - F(\overline{m}_i)}\Big] dF(v_i) + \sum_{i \in N} (\chi_i + \tau_i - \lambda) U_i(v_i^*) + \sum_{i \in N} \tau_i l_i.$$

Notice that

$$\int_{\overline{m}_i}^{\overline{v}} \big(S_i(\overline{m}_i) - r_i\big)\alpha(v_i \mid \lambda) dF(v_i) = \int_{\overline{m}_i}^{\overline{v}} \big(S_i(\overline{m}_i) - r_i\big)\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] dF(v_i), \tag{21}$$

Then, define the *virtual valuation* of agent i as

$$\Gamma_i(v_i \mid v_i^*, \overline{m}_i, \lambda) := \begin{cases} \beta(v_i \mid \lambda) & \text{if} \quad v_i \in [\underline{v}, v_i^*) \\ v_i^* & \text{if} \quad v_i = v_i^* \\ \alpha(v_i \mid \lambda) + \frac{\tau_i}{f(v_i)} & \text{if} \quad v_i \in (v_i^*, \overline{m}_i) \\ \mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \frac{\tau_i \overline{m}_i}{1 - F(\overline{m}_i)} & \text{if} \quad v_i \geqslant [\overline{m}_i, \overline{v}]. \end{cases}$$

Using the fact that $S_i(v_i) = \int_{V^{n-1}} s_i(v) \prod_{j \neq i} dF(v_j)$ and the definition of Γ_i, the Lagrangian simply rewrites

$$\mathcal{L} = \int_{V^n} (s_i(v) - r_i)\Gamma_i(v_i \mid v_i^*, \overline{m}_i, \lambda) \prod_{k \in N} dF(v_k) + \sum_{i \in N} (\chi_i + \tau_i - \lambda)U_i(v_i^*) + \sum_{i \in N} \tau_i l_i.$$

Notice that Γ_i is strictly increasing over $[\underline{v}, v_i^*)$ and $(v_i^*, \overline{m}_i)$ and constant over $[\overline{m}_i, \overline{v}]$. Yet, the virtual valuation Γ_i has a downward discontinuous jump at v_i^* and it is not clear how it behaves at $\overline{m}_i$. Therefore, it is not possible to directly solve the problem by pointwise maximizing $\mathcal{L}$ as it would violates the monotonicity of S_i at $v_i = v_i^*$.

To ensure that (IC1) is satisfied, I first replace the virtual valuation Γ_i by an ironed virtual valuation around v_i^* (see Myerson (1981), Lu and Robert (2001) and Loertscher and Wasser (2019)). Formally, for any given λ, define $\overline{x}_i \in V$ such that $\beta(\overline{x}_i \mid \lambda) = \alpha(\overline{m}_i \mid \lambda) + \frac{\tau_i}{f(\overline{m}_i)}$. Now, for any $x_i \in [\underline{v}, \overline{x}_i]$ let y_i be such that

$$\alpha(y_i \mid \lambda) + \frac{\tau_i}{f(y_i)} = \beta(x_i \mid \lambda). \tag{22}$$

Then, define

$$\delta_i(v_i \mid x_i, \overline{m}_i, \lambda) := \begin{cases} \Gamma_i(v_i \mid x_i, \overline{m}_i, \lambda) & \text{if } v_i \notin [x_i, y_i] \\ \beta(x_i \mid \lambda) & \text{if } v_i \in [x_i, y_i]. \end{cases}$$

The function $\delta_i(\cdot \mid \cdot)$ is referred to as the *ironed virtual valuation* of agent i. It coincides with the virtual valuation Γ_i everywhere but on $[x_i, y_i]$ where it is constant. By definition of x_i and y_i, $\delta_i(\cdot \mid \cdot)$ is therefore increasing and continuous on $[\underline{v}, \overline{m}_i]$.

The methodology is as follows. First, I replace Γ_i by δ_i in the Lagrangian of the relaxed problem and solve for s_i in the new problem by pointwise maximization. Building on Boulotov and Severinov (2018), I show that δ_i is nondecreasing at $v_i = \overline{m}_i$ so that, combined with the definition of δ_i for $v_i \notin [x_i, y_i]$, the solution s_i is increasing in v_i and so does S_i, satisfying (IC1). Second, I prove that the solution to the problem with ironed virtual valuation δ_i also solves the problem with virtual valuation Γ_i.

For some $x_i \in [\underline{v}, \overline{x}_i]$, consider the problem of maximizing the following Lagrangian

$$\hat{\mathcal{L}} := \int_{V^n} (s_i(v) - r_i)\delta_i(v_i \mid x_i, \overline{m}_i, \lambda) \prod_{k \in N} dF(v_k) + \sum_{i \in N} (\chi_i + \tau_i - \lambda)U_i(x_i) + \sum_{i \in N} \tau_i l_i.$$

Pointwise maximization with respect to s_i gives

$$s_i(v_i, v_{-i}) = \begin{cases} 1 & \text{if } \delta_i(v_i \mid x_i, \overline{m}_i, \lambda) > \max_{j \neq i} \delta_j(v_j \mid x_j, \overline{m}_j, \lambda) \\ \in [0,1] & \text{if } \delta_i(v_i \mid x_i, \overline{m}_i, \lambda) = \max_{j \neq i} \delta_j(v_j \mid x_j, \overline{m}_j, \lambda) \\ 0 & \text{if } \delta_i(v_i \mid x_i, \overline{m}_i, \lambda) < \max_{j \neq i} \delta_j(v_j \mid x_j, \overline{m}_j, \lambda), \end{cases} \tag{23}$$

as the problem is linear in s_i, full ownership goes to the agent with the highest ironed virtual valuation δ_i. If two or more agents have the highest valuation, the final distribution of ownership among those agents does not affect optimality. Yet, contrary to the first-best mechanism, ties may occur with positive probability (due to bunching regions) so that the way the mechanism breaks ties among agents will now affect IIR. As in Lu and Robert (2001) and Boulatov and Severinov (2019), the design of tie-breaking rules becomes an important element of the mechanism.

Using equation (23), the Lagrangian $\hat{\mathcal{L}}$ rewrites

$$\hat{\mathcal{L}} = \int_{V^n} \left\{ \max_{i \in N} \delta_i(v_i \mid x_i, \overline{m}_i, \lambda) - \sum_{i \in N} r_i \delta_i(v_i \mid x_i, \overline{m}_i, \lambda) \right\} \prod_{k \in N} dF(v_k) + \sum_{i \in N} (\chi_i + \tau_i - \lambda) U_i(x_i) + \sum_{i \in N} \tau_i l_i.$$

The following result is a generalization of Boulatov and Severinov (2018, Theorem 1, p. 15). Define first,

$$\delta_i^-(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) = \lim_{v_i \uparrow \overline{m}_i} \delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) = \alpha(\overline{m}_i \mid \lambda) + \frac{\tau_i}{f(\overline{m}_i)},$$

then,

Theorem 6 *A solution to the maximization of $\hat{\mathcal{L}}$ is such that*

1. *For all $i \in N$ such that $\overline{m}_i \leqslant \overline{m}_j$ for some $j \neq i$, $\delta_i(v_i \mid x_i, \overline{m}_i, \lambda)$ is continuous at $v_i = \overline{m}_i$, that is,*

$$\tau_i = \frac{\Big(\mathbb{E}\left[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\right] - \alpha(\overline{m}_i \mid \lambda)\Big)(1 - F(\overline{m}_i)) f(\overline{m}_i)}{1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)}. \tag{24}$$

2. *If it exists, agent $z_1 \in N$ such that $\max_{i \neq z_1} \overline{m}_i < \overline{m}_{z_1} < \overline{v}$ then $\delta_{z_1}(\overline{m}_{z_1} \mid x_{z_1}, \overline{m}_{z_1}, \lambda) > \delta_{z_1}^-(\overline{m}_{z_1} \mid x_{z_1}, \overline{m}_{z_1}, \lambda) = \max_{i \neq z_1} \delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda)$. Hence, we have*

$$\alpha(\overline{m}_{z_1} \mid \lambda) + \frac{\tau_{z_1}}{f(\overline{m}_{z_1})} = \max_{i \neq z_1} \frac{\int_{\overline{m}_i}^{\overline{v}} \alpha(v_i \mid \lambda) dF(v_i) + \overline{m}_i f(\overline{m}_i) \alpha(\overline{m}_i \mid \lambda)}{1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)}. \tag{25}$$

Proof. See Appendix B. ■

Although complex, Theorem 6 gives one important simple result: At $v_i = \overline{m}_i$, the ironed virtual valuation $\delta_i(\cdot \mid \cdot)$ is either continuous or has a discontinuous upward jump. Hence, combined with the fact that by definition $\delta_i(\cdot \mid \cdot)$ is increasing on $[\underline{v}, \overline{m}_i]$, the ironed virtual valuation is therefore nondecreasing for all $v_i \in V$. It follows that the allocation rule that maximizes $\hat{\mathcal{L}}$, equation (23), is nondecreasing in v_i. Thus, S_i is also nondecreasing in v_i and pointwise maximization of $\hat{\mathcal{L}}$ satisfies (IC1). Then the following holds.

Theorem 7 *If there exists an* $x := (x_1, \ldots, x_n) \in \times_{i \in N}[\underline{v}, \overline{x}_i]$ *and a solution to the following problem,*

$$\text{(A)} \quad \max_{s, \overline{m}, U, \tau, \chi} \quad \hat{\mathcal{L}} = \int_{V^n} \sum_{i \in N} (s_i(v) - r_i)\delta_i(v_i \mid x_i, \overline{m}_i, \lambda) \prod_{k \in N} dF(v_k) + \sum_{i \in N} (\chi_i + \tau_i - \lambda)U_i(x_i) + \sum_{i \in N} \tau_i l_i$$

$$\text{s.t.} \quad s_i(v) \geqslant 0 \quad \forall i \in N, v \in V^n$$

$$\sum_{j \in N} s_j(v) = 1,$$

$$\text{(B) For all } i \in N, \; S_i(v_i) = r_i \quad \text{for} \quad v_i \in [x_i, y_i].$$

Then if $(s^*, U^*, \overline{m}^*, \tau^*, \chi^*)$ *satisfies (A) and (B) it also solves the full problem.*

Proof. See Appendix B.

From Theorem 7, directly maximizing $\hat{\mathcal{L}}$ subject to the resource constraints is sufficient to obtain the solution to the general problem. So, the second-best solution amounts to give the final ownership to the agent with the highest ironed virtual valuation characterized by $\delta_i(v_i \mid \cdot)$. Those ironed virtual valuations are nondecreasing in v_i so that higher valuations give (weakly) better chances to receive final ownership. In case of tie (in terms of ironed virtual valuation), however, the second-best solution can lead to a situation in which more than one agent receives final ownership shares.

Yet, it remains to determine to characterize more precisely the ironed virtual valuations that depend on the endogenously determined cut-off types $\overline{m}$ and the Lagrange multipliers τ associated with (CC1). The following three corollaries derived from Theorem 6 help better characterize the ironed virtual valuations. The proofs are relegated to Appendix B.

Corollary 1 *At optimum,* τ_i *is decreasing in* $\overline{m}_i$ *for all* $i \in N$.

Corollary 2 *For all* $i \in N$ *such that* $\delta_i^-(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) = \delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda)$, $\delta_i(\overline{m}_i)$ *is increasing in* $\overline{m}_i$.

Corollary 3 *There is a bijection between* $\overline{m} = (\overline{m}_1, \ldots, \overline{m}_n)$ *and* $\tau = (\tau_1, \ldots, \tau_n)$ *according to equations* (24) *and* (25).

To illustrate the intuition of these results, take two agents with $\overline{m}_i \leqslant \overline{m}_j$, then $\tau_i \geqslant \tau_j$ from Corollary 1. From Corollary 2, $\delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) \leqslant \delta_j(\overline{m}_j \mid x_j, \overline{m}_j, \lambda)$. Loosely speaking, agent j is given and advantage over agent i in terms of ironed virtual valuation for large valuations as $\delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) \leqslant \delta_j(\overline{m}_j \mid x_j, \overline{m}_j, \lambda)$ but this advantage is compensated by a disadvantage for middle range valuations as $\tau_i \geqslant \tau_j$ implies that $\alpha(u \mid \lambda) + \frac{\tau_i}{f(u)} \geqslant \alpha(u \mid \lambda) + \frac{\tau_j}{f(u)}$ for some $u \in V$.

Corollary 3 shows that the relationship between the cut-off types $\overline{m}$ and the Lagrange multipliers τ is a one-to-one relationship. It follows that determining the optimal values of $\overline{m}$ (resp. τ) uniquely determines the optimal values of τ (resp. $\overline{m}$) therefore greatly simplifying the problem.

Moreover, differentiating $\hat{\mathcal{L}}$ with respect to $U_i(x_i)$ gives $\chi_i + \tau_i - \lambda = 0$. Hence, for any $\lambda > 0$ (*i.e.* nonnegative weight on collected revenue), it is clear that χ_i and τ_i cannot be simultaneously null, that is, either the individual rationality or the cash constraint (or both) is binding. As $\chi_i = \lambda - \tau_i$ for all $i \in N$ it is straightforward that if $\overline{m}_i \leqslant \overline{m}_j$ then $\chi_i \leqslant \chi_j$ as $\tau_i \geqslant \tau_j$. If some agent j with cut-off $\overline{m}_j$ gets some rent at optimum, *i.e.* $U_j(x_j) > 0$, all agents i with a lower cut-off $\overline{m}_i \leqslant \overline{m}_j$ must also get some rent, *i.e.* $U_i(x_i) > 0$ as $\chi_i \leqslant \chi_j = 0$ implies $\chi_i = 0$. Therefore, there is a clear relationship between the cut-off value $\overline{m}_i$ and the presence of rents.

In order to obtain a more detailed characterization of the second-best solution, the next subsection investigates the case of bilateral equal-share partnerships.

6.3. *Bilateral Equal-Share Partnerships*

Consider a partnership with $n = 2$ whose partners are denoted by i and j. To focus on the problem of cash-constraints, assume equal distribution of ownership shares $r_1 = r_2 = \frac{1}{2}$.

From Theorem 7, a solution to the second-best problem must be such that $S_i(v_i) = r_i$ for all $v_i \in [x_i, y_i]$ and $S_j(v_j) = r_j$ for all $v_j \in [x_j, y_j]$. In particular, $S_i(x_i) = S_j(x_j) = \frac{1}{2}$. Assume $x_i < x_j$, then $\delta_i(u) = \delta_j(u)$ for all $u \in [\underline{v}, x_i]$. It follows that $S_i(u) = S_j(u)$ for all $u \in [\underline{v}, x_i]$. But then as $S_i(x_i) = S_j(x_j) = \frac{1}{2}$ it must be that $S_j(w) = \frac{1}{2}$ for all $w \in [x_i, x_j]$ as S_j must be nondecreasing. Yet, for $S_j(w)$ to be constant on $w \in [x_i, x_j]$ it is necessary that $\delta_j(w)$ is also constant, however, $x_i < x_j$ implies that $\delta_j(w)$ is strictly increasing on $w \in [x_i, x_j]$. Hence, $x_i < x_j$ is not possible. The exact same reasoning applies for $x_i > x_j$. Therefore, the only possible solution is $x_i = x_j =: x^*$.

Now, it is possible to characterize the relationship between the cash resources l_i and l_j with the threshold values $\overline{m}_i$ and $\overline{m}_j$.

Lemma 5 *If* $l_i > l_j$ *then* $\overline{m}_i \geqslant \overline{m}_j$.

Proof. Assume that $l_i > l_j$ but assume instead that $\overline{m}_i < \overline{m}_j$. Then, $\tau_i > \tau_j$ and $y_i < y_j$ from the definition of y_i, equation (22). Moreover $\tau_i > \tau_j$ implies that $\tau_i > 0$ and thus (CC1) is binding for agent i. Moreover, $\tau_i > \tau_j$ implies that $\delta_i(u) \geqslant \delta_j(u)$ for all $u \in [y_i, \overline{m}_i]$ and thus $S_i(u) \geqslant S_j(u)$ for all $u \in [y_i, \overline{m}_i]$. From Corollary 2, $\delta_i(\overline{m}_i) < \delta_j(\overline{m}_j)$ and then $S_i(\overline{m}_i) \leqslant S_j(\overline{m}_j)$. Then, as (CC1) must be satisfied for agent j. Notice that as $\overline{m}_i < \overline{m}_j$ it must be also be that $\chi_i < \chi_j$ so that $\chi_j > 0$ and thus $U_j(x_j) = 0$. Therefore, (CC1) for agent j writes

$$\begin{aligned} l_j \geqslant& \overline{m}_j\Big(S_j(\overline{m}_j) - \frac{1}{2}\Big) - \int_{y_j}^{\overline{m}_j} \Big(S_j(u) - \frac{1}{2}\Big)du \\ =& \overline{m}_i\Big(S_j(\overline{m}_j) - \frac{1}{2}\Big) + \int_{\overline{m}_i}^{\overline{m}_j} \Big(S_j(\overline{m}_j) - S_j(u)\Big)du - \int_{y_j}^{\overline{m}_i} \Big(S_j(u) - \frac{1}{2}\Big)du \\ \geqslant& \overline{m}_i\Big(S_i(\overline{m}_i) - \frac{1}{2}\Big) - \int_{y_j}^{\overline{m}_i} \Big(S_j(u) - \frac{1}{2}\Big)du, \end{aligned}$$

where the third line stems from the fact that $S_i(\overline{m}_i) \leqslant S_j(\overline{m}_j)$ and, $\int_{\overline{m}_i}^{\overline{m}_j} \Big(S_j(\overline{m}_j) - S_j(u)\Big)du \geqslant 0$ as $\overline{m}_i < \overline{m}_j$ and $S_j(\overline{m}_j) \geqslant S_j(u)$ for all $u \in [\overline{m}_i, \overline{m}_j]$.

Assume first that $\overline{m}_i \leqslant y_j$, then $\int_{y_j}^{\overline{m}_i} \Big(S_j(u) - \frac{1}{2}\Big)du = 0$ as $S_j(u) = \frac{1}{2}$ for all $u \in [\overline{m}_i, y_j] \subset [x^*, y_j]$. Assume now that $\overline{m}_i > y_j$, then $\int_{y_j}^{\overline{m}_i} \Big(S_j(u) - \frac{1}{2}\Big)du \leqslant \int_{y_i}^{\overline{m}_i} \Big(S_i(u) - \frac{1}{2}\Big)du$ as $S_i(u) \geqslant S_j(u)$ for all $u \in [y_j, \overline{m}_i]$ and $S_i(u) - \frac{1}{2} \geqslant 0$ for all $u \in [y_i, y_j]$. Hence, in both cases,

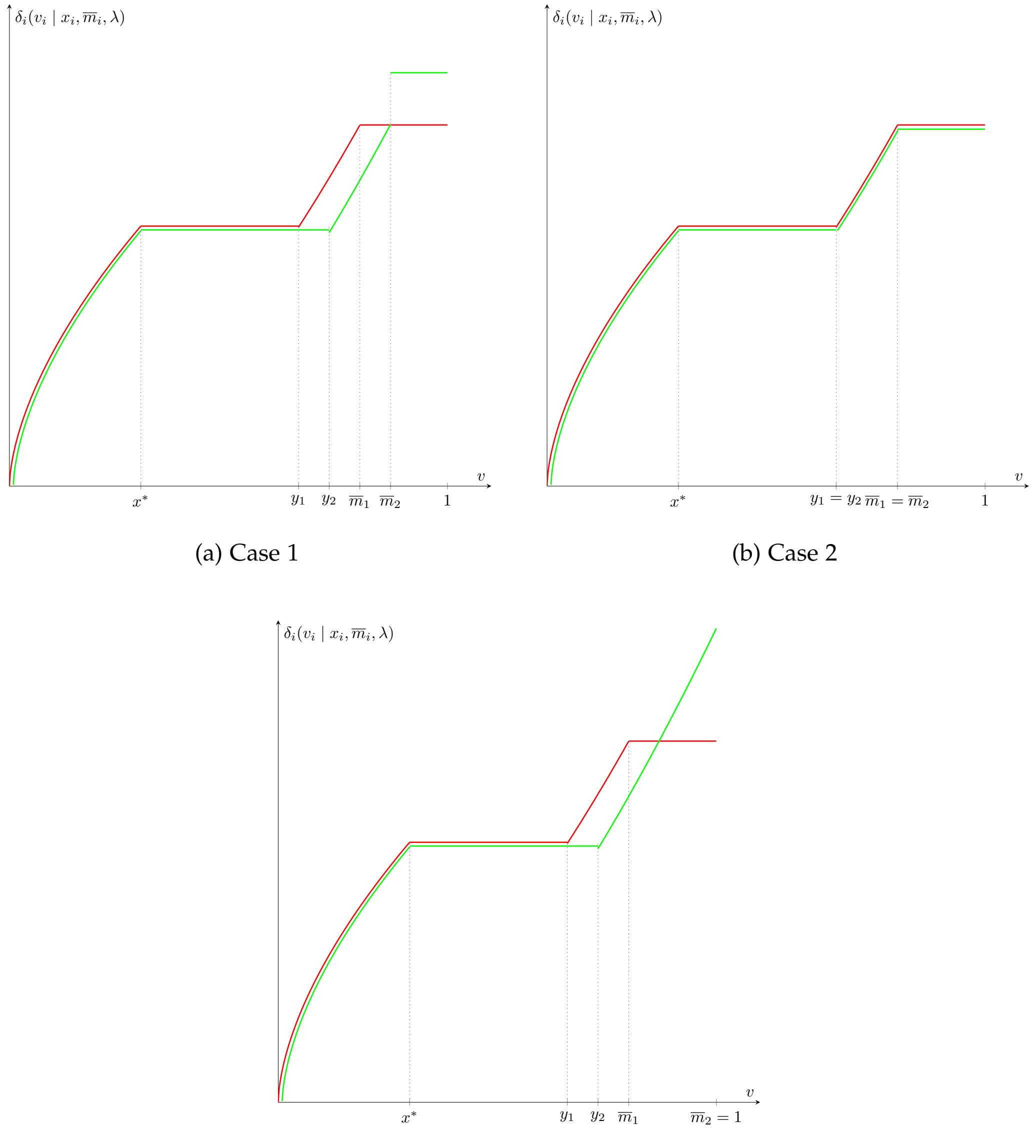
$$\begin{aligned} l_j \geqslant \overline{m}_i\Big(S_i(\overline{m}_i) - \frac{1}{2}\Big) - \int_{y_j}^{\overline{m}_i} \Big(S_j(u) - \frac{1}{2}\Big)du \geqslant& \overline{m}_i\Big(S_i(\overline{m}_i) - \frac{1}{2}\Big) - \int_{y_i}^{\overline{m}_i} \Big(S_i(u) - \frac{1}{2}\Big)du \\ =& U_i(x_i) + l_i, \end{aligned}$$

where the equality holds as (CC1) binds for agent i. From (IR), $U_i(x_i) \geqslant 0$ so that the above result implies that $l_i \leqslant l_j$, contradicting the initial assumption that $l_i > l_j$. Therefore, $l_i > l_j$ implies that $\overline{m}_i \geqslant \overline{m}_j$. ■

Assume now, that $l_2 > l_1$. It follows from Lemma 5 that $\overline{m}_1 \leqslant \overline{m}_2$ and thus $\tau_1 \geqslant \tau_2$. Figure 1 illustrates the different possible shapes for the ironed virtual valuations of each agent where it is assumed, without loss of generality, that $V = [0, 1]$ to simplify the exposition.

Case 1 illustrates the case in which both agents are cash-constrained but l_1 and l_2 are far apart. In that case, agent 2 is defined as in equation (25) of Theorem 6 and agent 1 by equation (24). Therefore agent 1 has a continuous ironed virtual valuation at $\overline{m}_1$ while agent 2's ironed virtual valuation jumps at $\overline{m}_2$. Both agents are similar for low and medium valuations (below y_1) but then agent 1 is advantaged over $[y_1, \overline{m}_1]$ and agent 2 is advantaged over $[\overline{m}_1, \overline{v}]$.

In case 2, the two agents share the same cut-off $\overline{m}_1 = \overline{m}_2$. This happens whenever they are both cash-constrained and the difference between l_1 and l_2 is small. As $\overline{m}_1 = \overline{m}_2$ it follows that τ_1 and τ_2 are defined by equation (24) so that both ironed virtual valuations are continuous at $\overline{m}_1 = \overline{m}_2$.



(a) Case 1

(b) Case 2

(c) Case 3

Figure 1: Ironed virtual valuations for agent 1 (red) and 2 (green) when $l_1 < l_2$.

Case 3 occurs when agent 1 is cash-constrained while agent 2 is almost not cash-constrained. In that case, agent 2's ironed virtual valuation is not distorted for large valuations but agent 2 is still disadvantaged over agent 1 for some valuations above y_1.

6.4. *Budget Balanced Second-Best Mechanisms*

Now, assume that the goal is to maximize the ex ante surplus subject to incentive compatibility, individual rationality, cash constraints *and* budget balance as in Section 3.[22] This problem can be written as

$$\max_{s,t} \sum_{i \in N} \mathbb{E}\Big[v_i(s_i(v) - r_i)\Big],$$

subject to incentive compatibility, individual rationality, cash constraints and $-\sum_{i\in N} \mathbb{E}\Big[t_i(v)\Big] \geqslant K$. Where $K \in \mathbb{R}_+$ is a given budget limit that should not be exceeded by the transfers made to the agents. Denote by $\lambda \geqslant 0$, the Lagrange multiplier associated with the budget balance constraint so that the problem rewrites,

$$\max_{s,t} \sum_{i \in N} \mathbb{E}\Big[v_i(s_i(v) - r_i)\Big] + \lambda\Big(- \sum_{i \in N} \mathbb{E}\Big[t_i(v)\Big] - K\Big),$$

subject to incentive compatibility, individual rationality and cash constraints. Notice that maximizing this objective is equivalent to maximize the following objective

$$\max_{s,t} \quad (1+\lambda) W_{\frac{\lambda}{1+\lambda}}(s,t) - \lambda K,$$

subject to incentive compatibility, individual rationality and cash constraints and where W_ω is given by equation (18), *i.e.* the objective function of the problem solved in Section 6, where λ is replaced by $\frac{\lambda}{1+\lambda}$ and it is clear that $\frac{\lambda}{1+\lambda} \in [0, 1)$.

Therefore, maximizing the ex ante surplus subject to incentive compatibility, individual rationality, cash constraints *and* budget balancedness is equivalent to solve the second-best problem derived in Subsection 6.2 for a well chosen λ.

7. EXTENSIONS

7.1. *Other Ownership Structures*

The ex post efficient dissolution condition, the Equivalence Theorem and the auction (Theorems 1, 2, 3, 4 and 5) actually apply to more general settings than the partnership problem discussed above. The interpretation of the setting as a partnership problem lies in the assumption that

[22] Although, as previously said, I restrict the analysis to ex ante budget balanced mechanisms rather than ex post budget balanced due to the complexity of the latter.

for all $i \in N$, $r_i \in [0,1]$ and $\sum_{i \in N} r_i = 1$, that is, each partner initially owns a share of an asset which is then redistributed to the one with the highest valuation.

Although crucial for the characterization of optimal initial ownership structures in Section (4) and their interpretation, this assumption plays a very limited role in the derivation of Theorems 1-5. Coming back to Section 3, recall that IIR can be written as $H_i \leqslant C(r_i)$ where $C(r_i) := \inf_{v_i} \{\mathbb{E}_{-i} g(v) - v_i r_i\}$. From there on, Theorems 1-5 can be written only using the definition of $C(r_i)$ without any particular reference to the particular outside options $v_i r_i$.

Consider the following. If agent $i \in N$ refuses to participate in the mechanism, then it receives $\varphi_i(v;\theta,l) \in [0,\bar{v}]$ where φ_i is assumed to be concave in v_i and $\theta \in \mathbb{R}^n_+$ is some vector of parameters where $\theta_i \in \mathbb{R}_+$ is associated with agent i. Notice that φ_i may depend on all valuations, on the cash resources and on some other parameters defining the agents. Setting $\varphi_i(v;\theta,l) := v_i\theta_i$ with $\sum_{i \in N} \theta_i = 1$ simply replicates the ownership structure investigated in the rest of the paper. In the general case, IIR writes

$$v_i S_i^*(v_i) + T_i^*(v_i) - \mathbb{E}_{-i}\varphi_i(v;\theta,l) \geqslant 0, \text{ for all } v_i \in V, i \in N.$$

Replacing $T_i^*(v_i) := \mathbb{E}_{-i} t_i^*(v)$ where $t_i^*(v)$ is given by equation (2), IIR rewrites

$$H_i \leqslant \mathbb{E}_{-i} g(v) - \mathbb{E}_{-i}\varphi_i(v;\theta,l), \text{ for all } v_i \in V, i \in N.$$

Define for each $i \in N$,

$$Z_i(\theta,l) := \inf_{v_i \in V} \{\mathbb{E}_{-i} g(v) - \mathbb{E}_{-i}\varphi_i(v;\theta,l)\},$$

then IIR rewrites

$$H_i \leqslant Z_i(\theta,l), \text{ for all } i \in N.$$

The function $Z_i(\cdot)$ serves the same role as $C(\cdot)$ in Section 3. It defines the maximal amount that can be collected on agent i without violating their interim individual rationality constraint. Hence, $v_i^* \in \arg\min_{v_i} \{\mathbb{E}_{-i} g(v) - \mathbb{E}_{-i}\varphi_i(v;\theta,l)\}$ is agent i's worst-off type which is characterized by the first-order condition of the minimization problem: $F(v_i^*)^{n-1} = \frac{\partial}{\partial v_i}\mathbb{E}_{-i}\varphi_i(v;\theta,l)$.[23]

The condition for the existence of ex post efficient dissolution (with Bayesian or Dominant Strategy mechanisms) then simply rewrites

$$\sum_{i \in N} \min\{Z_i(\theta,l), l_i\} \geqslant B.$$

[23]The first-order condition is necessary and sufficient. Indeed the second-order derivative of $\mathbb{E}_{-i} g(v) - \mathbb{E}_{-i}\varphi_i(v;\theta,l)$ immediately writes $(n-1)f(v_i)F(v_i)^{n-2} - \frac{\partial^2}{\partial v_i^2}\mathbb{E}_{-i}\varphi_i(v;\theta,l) < 0$ as $\varphi_i(v;\theta,l)$ is assumed to be concave in v_i.

The equivalence theorems still hold and the cash-constrained auction (Section 5) still applies by modifying the side payments accordingly. Therefore, the dissolution condition applies to more general settings than the ones with outside options of the form $v_i r_i$.

Example 1: Allocation of a Private Good. The simplest and most general extension of the partnership dissolution framework is that of the allocation of a private good to agents with type-dependent outside opportunities. Assume the state wants to privatize a publicly-owned asset (road, spectrum, ...). A group of n candidate firms is considered. None of the firms has initial ownership rights in the publicly-owned asset but each has some outside opportunities that likely depend on their ability to efficiently run a business. Applying the partnership dissolution framework with general outside opportunities $\varphi_i(\cdot)$ directly provides the mechanism that efficiently allocates the asset to the most efficient firm. Those kinds of allocation problems generally involve highly valued assets and require large monetary transfers to get control rights. As previously showed, the mechanism I propose performs better than that of CGK in the presence of limited cash resources and could be used to help small firms with low cash resources to compete against larger well-established incumbents.

Example 2: Silent Partners. This example is inspired by Ornelas and Turner (2007). Assume n partners jointly own a business where each of them can claim a share $r_i \in [0,1]$ (and $\sum_{i\in N} r_i = 1$) of total output value. Each partner's valuation $v_i \in V$ represents their ability to run the business and the business value is given by the valuation of the partner in charge, say partner 1. In this setting, it is assumed that partner 1 has claims r_1 on total output value but is also given *control rights* over the business. This framework provides a way to unbundle *control rights* from *ownership rights* on the value of the output.

Let (with a slight abuse of notations) $\varphi_1 := v_1 r_1$ and $\varphi_j := \mathbb{E}_{-j} v_1 r_j$ for all $j \neq 1$. The difference with the standard case is that if agent $i \in N$ refuses to participate in the mechanism, the business continues as usual, that is, it is still worth v_1.

Assume now that the partners consider dissolving their partnership and let $Z_i(r_i)$ be partner i's maximal payment before refusing to participate in dissolution. Then $Z_1(r_1) = C(r_1)$ as partner 1 exhibits exactly the same outside option as in the previous analysis. On the contrary, for each $j \neq 1$, $Z_j(r_j) = \inf_{v_j} \left\{\mathbb{E}_{-j} g(v) - r_j \mathbb{E}_{-j} v_1\right\} = \int_{\underline{v}}^{\overline{v}} x dF(x)^{n-1} - r_j \int_{\underline{v}}^{\overline{v}} x dF(x)$ where it is easy to show that the worst-off type is $v_j^* = \underline{v}$ for all $j \neq 1$. The dissolution condition therefore writes $\sum_{i\in N} \min\{Z_i(r_i), l_i\} \geqslant B$.

Let $n = 3$, $F(v_i) = v_i$ and $V = [0,1]$. Then $Z_1(r_1) = 2/3 - (2/3) r_1^{3/2}$ and $Z_j(r_j) = 2/3 - (1/2) r_j$ for $j \neq 1$. In the absence of cash constraints, dissolution is possible if and only if $r_1 \leqslant 9/16$ and the remaining shares can be distributed arbitrarily between the silent partners. Furthermore, the extreme ownership structure where $r_1 = 0$ is also dissolvable. With cash-constraints, however, those extreme ownership structures where r_1 is close to zero will generally prevent ex post efficient dissolution. Assume for instance that partner 1 has $l_1 = 1/2$ while the other two partners are not cash-constrained. Then, ex post efficient dissolution is impossible whenever $r_1 < 1/3$. Cash constraints somehow prevents extreme ownership structures to be desirable in some cases.

7.2. *Asymmetric Distributions of Valuations*

Most of the results concerning ex post efficient dissolution mechanisms naturally extends to asymmetric distribution of the agents' valuations. More precisely, Theorems 1, 2, 3 and 4 still hold with only slight modifications of the dissolution condition for the first two. I give the main elements of the proof for Bayesian mechanisms which mainly consists in the construction of the transfer function to satisfy EPCC. The case of Dominant Strategy mechanisms can easily be obtained using this transfer function and the proof of Theorem 2 and is therefore omitted.

Extending the analysis to asymmetric distributions of valuations is important as many applications of the partnership dissolution framework can be better represented that way. In the divorce problem for instance, the asset that has to be reallocated might be the family house of one of the spouses. The family house might have a low market value but a large sentimental value for the spouse who has spent their childhood in the family house. Therefore it is more likely that this spouse has a larger valuation for the family house than the other one, *i.e.* distributions of valuations are likely to be asymmetric. In the biotechnology sector, large well-established firms partner with small young firms to develop new research. It is more likely that large firms have higher valuations (as they are more capable of using the results of the research) than small firms. The question is therefore to understand how asymmetric distributions of valuations and limited cash resource interplay and affect the dissolution condition.

THE DISSOLUTION CONDITION. Consider the same setting as the one described in Section 2 except for the distribution of valuations. Assume that partner $i \in N$ has valuation $v_i \in V$ where each v_i is independently distributed from an absolutely continuous cumulative distribution function F_i. Let $f_i = F_i'$ be the probability distribution function for v_i. For convenience for any $x \in V$, let $\mathbf{G}(x) := \prod_{k \in N} F_k(x)$, $\mathbf{G}_i(x) := \prod_{k \neq i} F_k(x)$ and $\mathbf{G}_{ij}(x) := \prod_{k \neq i,j} F_k(x)$ denote the distributions of the maximum of all valuations, all valuations except v_i and all valuations except v_i and v_j, respectively.

Proposition 1 holds with asymmetric distributions of valuations (see Makowski and Mezzetti, 1994), then the transfer function of partner $i \in N$ must write $t_i(v) = g(v) - v_i s_i^*(v) - h_i(v)$. Recall that the ex ante cost of implementing a Groves mechanism writes $G := \mathbb{E}g(v) = \int_V x d\mathbf{G}(x)$ as in the case of symmetric distributions of valuations. The main difference concerns the upper bound of how much can be taken from partner i without violating their individual rationality constraints. In Section 2, this upper bound was defined by $C(r_i) = \inf_{v_i} \{\mathbb{E}_{-i} g(v) - v_i r_i\}$ so that two partners with the same initial ownership shares $r_i = r_j$ had the same upper bounds $C(r_i) = C(r_j)$. With asymmetric distributions of valuations two agents with the same initial ownership shares may have different upper bounds. Formally, simply let $\mathcal{C}_i(r_i) := \inf_{v_i}\left\{\mathbb{E}_{-i}g(v) - v_i r_i\right\}$ denote this upper bound in the asymmetric case. The only difference is that now each partner may have a different function $\mathcal{C}_i(\cdot)$ depending on their cumulative distribution function F_i. These new pieces of notations are sufficient to state the result extending the analysis to asymmetric distributions of valuations.

Proposition 6 *An EF, IIC (resp. EPIC), IIR, EPBB (resp. EABB) and EPCC dissolution mechanism exists **if and only if***

$$\sum_{i\in N}\min\{\mathcal{C}_i(r_i), l_i\} \geqslant (n-1)G. \tag{26}$$

Proof. I provide a sketch of the main elements of the proof for Bayesian mechanisms. See Appendix C for further details. The "only if" part (necessity) is obtained exactly as in the proof of Theorem 1: Simply replace $C(r_i)$ by $\mathcal{C}_i(r_i)$ in equation (6) and then combining it with (7) and (4) it gives (26).

The "if" part (sufficiency) requires the construction of an appropriate transfer function. Again, using the transfer function proposed by Dudek, Kim and Ledyard (1995) is helpful for the cash-constrained case. Consider the following:

$$t_i(v) := \begin{cases} -\displaystyle\int_{\underline{v}}^{v_i} \sum_{k\neq i} \frac{\int_{\underline{v}}^{x}\mathbf{G}(y)dy}{\mathbf{G}(x)}\frac{f_k(x)}{F_k(x)}dx - \frac{n-1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = i \\ \displaystyle\int_{\underline{v}}^{v_j} \frac{\int_{\underline{v}}^{x}\mathbf{G}(y)dy}{\mathbf{G}(x)}\frac{f_i(x)}{F_i(x)}dx + \frac{1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = j \neq i, \end{cases} \tag{27}$$

where $\rho(v)$ is defined as in Section 3 and $\phi_i \in \mathbb{R}$ is a constant.

It is immediate to see from (27) that EPBB is equivalent to $\sum_{i\in N}\phi_i = 0$. It is also EF as it allocates all the ownership shares to the partner with the highest valuation. Moreover, it is clear that the minimum of this function is attained when $\rho(v) = i$ and $v_i = \overline{v}$, that is

$$\min_{v\in V^n} t_i(v) = \phi_i - \int_{\underline{v}}^{\overline{v}} \sum_{k\neq i} \frac{\int_{\underline{v}}^{x}\mathbf{G}(y)dy}{\mathbf{G}(x)}\frac{f_k(x)}{F_k(x)}dx - \frac{n-1}{n}\underline{v}.$$

A little algebra on the integral term (see Appendix C for detailed computations) shows that EPCC can be written as

$$\phi_i \leqslant \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x}\mathbf{G}(y)dy}{\mathbf{G}(x)}\frac{f_i(x)}{F_i(x)}dx + \frac{1}{n}\underline{v} - G + l_i. \tag{28}$$

Computing the interim transfer $T_i(v_i) := \mathbb{E}_{-i}t_i(v)$ gives

$$T_i(v_i) = \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x}\mathbf{G}(y)dy}{\mathbf{G}(x)}\frac{f_i(x)}{F_i(x)}dx + \frac{1}{n}\underline{v} - G + \int_{v_i}^{\overline{v}} x\,d\mathbf{G}_i(x) - \phi_i,$$

Detailed computations of $T_i(v_i)$ and proof that the mechanism is IIC can be found in Appendix C. The characterization of $C(r_i) = \int_{v_i^*(r_i)}^{\overline{v}} x dF(x)^{n-1}$ given by equation (15) naturally extends to asymmetric distributions of valuations as follows: $\mathcal{C}_i(r_i) = \int_{v_i^*(r_i)}^{\overline{v}} x d\mathbf{G}_i(x)$, where $v_i^*(r_i)$ is the worst-off type that solves $\inf_{v_i} \{\mathbb{E}_{-i} g(v) - v_i r_i\}$ which is given by $\mathbf{G}_i(v_i^*(r_i)) = r_i$. As in CGK, satisfying IIR is equivalent to $T_i(v_i^*(r_i)) \geqslant 0$. Therefore, using the expression of $\mathcal{C}_i(r_i)$, IIR can be written as

$$\phi_i \leqslant \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x} \mathbf{G}(y)dy}{\mathbf{G}(x)} \frac{f_i(x)}{F_i(x)} dx + \frac{1}{n}\underline{v} - G + \mathcal{C}_i(r_i). \tag{29}$$

Combining equations (28) and (29) and simplifying gives

$$\phi_i \leqslant \min\{\mathcal{C}_i(r_i), l_i\} + \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x} \mathbf{G}(y)dy}{\mathbf{G}(x)} \frac{f_i(x)}{F_i(x)} dx + \frac{1}{n}\underline{v} - G. \tag{30}$$

Simply let

$$\phi_i = \min\{\mathcal{C}_i(r_i), l_i\} + \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x} \mathbf{G}(y)dy}{\mathbf{G}(x)} \frac{f_i(x)}{F_i(x)} dx + \frac{1}{n}\underline{v} - G - \frac{1}{n}\left[\sum_{j\in N} \min\{\mathcal{C}_j(r_j), l_j\} - (n-1)G\right].$$

Given that equation (26) is satisfied, it is then immediate that ϕ_i satisfies equation (30) so that the mechanism is IIR and EPCC. For EPBB, it is easy to show that $\sum_i \phi_i = 0$ holds after noticing that $\sum_{i\in N} \int_{\underline{v}}^{\overline{v}} \frac{\int_{\underline{v}}^{x} \mathbf{G}(y)dy}{\mathbf{G}(x)} \frac{f_i(x)}{F_i(x)} dx = G - \underline{v}$ where the result is obtained by integration by parts. ■

Some Characterization Results. Providing a full characterization of the dissolution condition with asymmetric distributions of valuations and asymetric cash resources is not very insightful. Many subcases can occur depending on the various initial conditions for distributions of valuations and for cash resources. Instead, I provide particular characterization results to illustrate some of the effects of asymmetric distributions of valuations on optimal initial ownership structures.

In order to make meaningful and easy comparisons between the partners, assume that distributions of valuations can be ranked according to first-order stochastic dominance as follows: For all $x \in V$, $F_1(x) \leqslant F_2(x) \leqslant \cdots \leqslant F_n(x)$. It follows that for any $i, j \in N$ with $i < j$, partner i is more likely to have a higher valuation than partner j. For any $i < j$ and $x \in V$, notice that $\mathbf{G}_i(x) = \mathbf{G}_j(x)\frac{F_j(x)}{F_i(x)} \geqslant \mathbf{G}_j(x)$ as $F_i(x) \leqslant F_j(x)$. Given that for any $k \in N$ and $\hat{r} \in [0,1]$, $\mathbf{G}_k(v_k^*(\hat{r})) = \hat{r}$ then for any i, j with $i < j$, $v_i^*(\hat{r}) \leqslant v_j^*(\hat{r})$ as $G_i(x) \geqslant G_j(x)$ and $G_k(x)$ is increasing for any $x \in V$. It follows that for any i, j with $i < j$ and any $\hat{r} \in [0,1]$, $\mathcal{C}_i(\hat{r}) \leqslant \mathcal{C}_j(\hat{r})$

as

$$\begin{aligned}\mathcal{C}_i(\hat{r}) - \mathcal{C}_j(\hat{r}) &= \int_{v_i^*(\hat{r})}^{\overline{v}} x d\mathbf{G}_i(x) - \int_{v_j^*(\hat{r})}^{\overline{v}} x d\mathbf{G}_j(x) \\ &= v_j^*(\hat{r})\mathbf{G}_j(v_j^*(\hat{r})) - v_i^*(\hat{r})\mathbf{G}_i(v_i^*(\hat{r})) - \int_{v_i^*(\hat{r})}^{\overline{v}} \mathbf{G}_i(x)dx + \int_{v_j^*(\hat{r})}^{\overline{v}} \mathbf{G}_j(x)dx \\ &= \int_{v_j^*(\hat{r})}^{\overline{v}} [\mathbf{G}_j(x) - \mathbf{G}_i(x)]dx - \int_{v_i^*(\hat{r})}^{v_j^*(\hat{r})} [\mathbf{G}_i(x) - \hat{r}]dx \leqslant 0,\end{aligned}$$

where the second line is obtained after integrating the two terms by parts and the third line uses the fact that $\mathbf{G}_i(v_i^*(\hat{r})) = \mathbf{G}_j(v_j^*(\hat{r})) = \hat{r}$. The inequality follows from $\mathbf{G}_i(x) \geqslant \mathbf{G}_j(x)$ for all $x \in V$, $v_i^*(\hat{r}) \leqslant v_j^*(\hat{r})$ and $\mathbf{G}_i(x) \geqslant \hat{r}$ for $x \in [v_i^*(\hat{r}), v_j^*(\hat{r})]$ so that the first is always nonpositive and the second one always nonnegative.

Consider now the optimal ownership structure $r^* \in \arg\max_{r \in \Delta^{n-1}} \sum_{i \in N} \mathcal{C}_i(r_i)$ in the absence of cash constraints. Recall that from the Envelope Theorem, $\mathcal{C}_k'(r_k) = -v_k^*(r_k)$, then optimality conditions implies $\mathcal{C}_i'(r_i^*) = \mathcal{C}_j'(r_j^*)$ for all i, j, which is equivalent to $v_i^*(r_i^*) = v_j^*(r_j^*)$. As $v_i^*(r_i^*) \leqslant v_j^*(r_j^*)$ and $v_k^*(\cdot)$ is increasing for all $k \in N$, it follows that the initial ownership structure maximizing $\sum_{i \in N} \mathcal{C}_i(r_i)$ must satisfy $r_1^* \geqslant r_2^* \geqslant \cdots \geqslant r_n^*$. This result is the one obtained by Figueroa and Skreta (2012, Corollary 1).

Intuitively, $\mathcal{C}_i(\hat{r}) \leqslant \mathcal{C}_j(\hat{r})$ for $i < j$ means that less money can be collected on partners whose valuations are more likely to be high as they are more likely to have a higher initial outside option. Optimality conditions reveal that it is better to give more initial ownership rights to those partners as already few money can be collected on them and fewer initial ownership rights to partners who are more likely to have low valuations in order to collect larger amount of money from them. This feature is reminiscent of the results of Figueroa and Skreta (2012) who characterize optimal ownership structures in partnership dissolution problems with asymmetric distributions of valuations. They show that if asymmetries in distributions of valuations are quite important, the optimal ownership structures might be very extreme. Limited cash resources, however, may mitigate those extreme ownership structures as it is illustrated in the following example.

A TWO-AGENT EXAMPLE. Take the case of a large pharmaceutical firm i and a small R&D firm j forming an alliance to develop a new drug. Ownership shares r_i and $r_j := 1 - r_i$ represent initial claims on the output generated with the new drug. It is reasonable to think that the pharmaceutical firm has a higher potential than the R&D firm to distribute the drug once it has been developed. This idea is simply modeled by assuming that the pharmaceutical firm is more likely to have a higher valuation for the drug than the R&D firm. Then, let $F_i(x) \leqslant F_j(x)$ for all $x \in V$. Furthermore, the small R&D firm can be assumed to be financially constrained while I assume that the large firm is not for convenience. Let $l_i = +\infty$ and $l_j < +\infty$ and define $r_i^* \in \arg\max_{r_i \in [0,1]} \mathcal{C}_i(r_i) + \mathcal{C}_j(1 - r_i)$.

Consider first the case in which $l_j \geqslant \mathcal{C}_j(1-r_i^*)$. Then, it is clear that r_i^* is also the solution to $\max_{\hat{r}_i} \mathcal{C}_i(\hat{r}_i)+\min\{\mathcal{C}_j(1-\hat{r}_i), l_j\}$. Previous computations show that $r_i^* > r_j^*$, *i.e.* the pharmaceutical firm should have more initial ownership rights as it is more likely to be the final owner of the drug. If asymmetries in distributions of valuations are large, then it can become optimal to initially give a very large share of ownership rights to the pharmaceutical firm.

But now consider the case in which $l_j < \mathcal{C}_j(1-r_i^*)$. It is clear that r_i^* is not the optimal ownership structure anymore. Indeed, take $\hat{r}_i$ such that $\mathcal{C}_j(1-\hat{r}_i) = l_j$. As $\mathcal{C}_j(\cdot)$ is a decreasing function this implies that $\hat{r}_i < r_i^*$ and as $\mathcal{C}_i(\cdot)$ is also a decreasing function it implies that $\mathcal{C}_i(\hat{r}_i) > \mathcal{C}_i(r_i^*)$. It follows that $\mathcal{C}_i(\hat{r}_i)+\mathcal{C}_j(1-\hat{r}_i) = \mathcal{C}_i(\hat{r}_i)+l_j > \mathcal{C}_i(r_i^*)+l_j$. If it were beneficial to further decrease $\hat{r}_i$ then the left derivative of $\mathcal{C}_i(r_i)+\mathcal{C}_j(1-r_i)$ at $r_i = \hat{r}_i$ would be negative, *i.e.*, $\mathcal{C}_i'(\hat{r}_i)-\mathcal{C}_j'(1-\hat{r}_i) < 0$. But from optimality conditions recall that $\mathcal{C}_i'(r_i^*) = \mathcal{C}_j'(1-r_i^*)$ and thus $\mathcal{C}_i'(\hat{r}_i) > \mathcal{C}_i'(r_i^*) = C_j'(1-r_i^*) > \mathcal{C}_j'(1-\hat{r}_i)$ as $\mathcal{C}_i'(\cdot)$ and $\mathcal{C}_j'(\cdot)$ are both decreasing. Therefore, the left derivative of the objective at $r_i = \hat{r}_i$ is nonnegative and $\hat{r}_i$ is the solution to the problem when $l_j < \mathcal{C}_j(1-r_i^*)$.

In the latter case, $\hat{r}_i < r_i^*$, that is, the distortion in initial ownership rights due to asymmetries in distributions of valuations is mitigated by the presence of cash constraints. Giving more initial ownership shares to the pharmaceutical firm is good as it is likely that it will efficiently be the final owner of the drug but at the same time it could make impossible an efficient buyout by the financially-constrained R&D firm.

8. CONCLUSION

In this paper, I study partnership dissolution problems with cash-constrained agents. This framework applies to various economic settings such as divorces, terminations of joint ventures, bankruptcy procedures or land reallocation. Relying on the mechanism design literature, I construct dissolution mechanisms that perform well even in the presence of cash-constrained agents. I derive necessary and sufficient conditions for ex post efficient partnership dissolution with interim (resp. ex post) incentive compatible, interim individually rational, ex post (resp. ex ante) budget balance and ex post cash-constrained mechanisms. I show that the dissolution condition is a generalization of the condition found in CGK. Interestingly, when partners have asymmetric cash constraints, the equal-share partnership is no more the *optimal* initial ownership structure (as found by CGK). Instead, the *optimal* initial ownership structure allocates relatively more (resp. less) property rights to more (resp. less) cash-constrained partners. This result sheds light on the role of the distributions of liquid and illiquid assets in organizations.

I show that the standard equivalence between Bayesian and dominant strategy mechanisms remains valid under the assumption of cash-constrained partners. This result indicates that both classes of mechanisms can be equivalently implemented and that there is no new restrictions due to the presence of cash-constraints.

I propose a simple "cash-constrained" auction to implement the ex post efficient dissolution mechanisms. It simply consists in asking agents to submit bids to an auctioneer who finally allocates ownership rights to the highest bidder. Prices and side payments are designed to satisfy all the desired properties of the dissolution mechanism. I further show that the cash-constrained auction allows to dissolve some partnerships with cash-constrained agents that CGK's auction would fail to dissolve.

Finally, I investigate second-best mechanisms whose objective function is a convex combination of the expected surplus and the expected collected revenues. I characterize the set of interim incentive compatible, interim individually rational and ex post cash-constrained mechanism for any allocation rule. I show that the problem can be solved using a relaxed problem. The solution involves (i) ironing of the virtual valuations of the partners and, (ii) favors heavily cash-constrained agents towards less cash-constrained agents for medium-range valuation and (iii) favors less cash-constrained agents towards heavily cash-constrained agents for high-range valuations. Imposing a budget balance condition can be simply done by solving the above problem for a well-chosen weight of the convex combination the expected surplus and the expected collected revenues.

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APPENDIX A

Additional computations for Theorem 1. To prove that t_i^C is IIC, Proposition 1 requires that $\mathbb{E}_{-i}h_i^C(v) = H_i$ where $H_i \in \mathbb{R}$ is a constant. Taking expectations $\mathbb{E}_{-i}$ over equation (9) gives

$$\begin{aligned}\mathbb{E}_{-i}h_i^C(v) &= \frac{n-1}{n}\left[\mathbb{E}_{-i}g(v) + \frac{1}{n-1}\sum_{j\neq i}\mathbb{E}_{-i}s_j(v)\psi(v_j) - \psi(v_i)\mathbb{E}_{-i}s_i(v)\right] + \phi_i \\ &= \frac{n-1}{n}\left[v_iF(v_i)^{n-1} + \int_{v_i}^{\overline{v}} x dF(x)^{n-1} + \int_{v_i}^{\overline{v}} \psi(v_j)F(v_j)^{n-2}dF(v_j) - \psi(v_i)F(v_i)^{n-1}\right] + \phi_i.\end{aligned}$$

Replacing $\psi(\cdot)$ by its expression and integrating the third term by part gives

$$\begin{aligned}\mathbb{E}_{-i}h_i^C(v) &= \frac{n-1}{n}\Bigg[v_iF(v_i)^{n-1} + \int_{v_i}^{\overline{v}} x dF(x)^{n-1} + \left(-\int_{\underline{v}}^{\overline{v}} F(x)^n dx + \frac{\int_{\underline{v}}^{v_i} F(x)^n dx}{F(v_i)} + \int_{v_i}^{\overline{v}} F(x)^{n-1}dx\right) \\ &\qquad - \frac{\int_{\underline{v}}^{v_i} F(x)^n dx}{F(v_i)}\Bigg] + \phi_i \\ &= \frac{n-1}{n}\left[v_iF(v_i)^{n-1} + \int_{v_i}^{\overline{v}} x dF(x)^{n-1} - \int_{\underline{v}}^{\overline{v}} F(x)^n dx + \int_{v_i}^{\overline{v}} F(x)^{n-1}dx\right] + \phi_i.\end{aligned}$$

Notice that, by integration by parts

$$\int_{v_i}^{\overline{v}} F(x)^{n-1}dx = \overline{v} - v_iF(v_i)^{n-1} - \int_{v_i}^{\overline{v}} x dF(x)^{n-1}.$$

Plugging this expression into $\mathbb{E}_{-i}h_i^C(v)$ gives

$$\begin{aligned}\mathbb{E}_{-i}h_i(v) &= \frac{n-1}{n}\left[\overline{v} - \int_{\underline{v}}^{\overline{v}} F(s)^n ds\right] - \phi_i \\ &= \frac{n-1}{n}G - \phi_i,\end{aligned}$$

which concludes the proof. ■

Proof of Lemma 1. Notice that,

$$\begin{aligned}\int_{\underline{v}}^{v_k} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt &= -\left[\frac{\int_{\underline{v}}^{t} F(s)^n ds}{nF(t)^n}\right]_{\underline{v}}^{v_k} + \frac{1}{n}\int_{\underline{v}}^{v_k} dt \\ &= \frac{1}{n}\left[v_k - \frac{\int_{\underline{v}}^{v_k} F(s)^n ds}{F(v_k)^n} - \underline{v}\right] \\ &= \frac{1}{n}\left[v_k - \psi(v_k) - \underline{v}\right],\end{aligned}$$

where the first line stems from integration by parts and the second line is obtained using L'Hôpital's rule, *i.e.*, $\lim_{t\to\underline{v}} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{nF(t)^n} = \lim_{t\to\underline{v}} \frac{F(t)}{nf(t)} = 0$.

Then, $v_k - \psi(v_k) = n\int_{\underline{v}}^{v_k} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \underline{v}$ from which it immediately follows that $[v_k - \psi(v_k)]$ is nonnegative and increasing in $v_k \in V$. ■

Proof of Theorem 2. (Necessity) Take a dissolution mechanism (s^*, t^*) satisfying EF and EPIC, *i.e.*, satisfying Proposition 1.b. Then, $t_i^*(v) = g(v) - v_is_i * (v_i) - h_i(v)$ with $h_i(v)$ is constant in v_i. Define $H_i := \mathbb{E}_{-i}h_i(v)$, it is then straightforward to see that equations (6) and (7) necessary in Bayesian mechanisms are also necessary for dominant strategy mechanisms. As shown in the proof of Theorem 1, the necessity of equations (6) and (7) implies the necessity of $\sum_{i\in N} \min\{C(r_i), l_i\} \geqslant (n-1)G$ which is (14).

(Sufficiency) Consider the following transfer function for agent i: $t_i^*(v) = g(v) - v_i s_i^*(v) - \Phi_i$ where $\Phi_i \in \mathbb{R}$ is a constant. From Proposition 1.b., $t_i^*(v)$ is EF and EPIC. EABB requires $\mathbb{E}\left[\sum_i t_i^*(v)\right] = 0$, which is equivalent to $\sum_i \Phi_i = (n-1)G$. EPCC requires that $t_i^*(v) = g(v) - v_i s_i^*(v) - \Phi_i \geqslant -l_i$ which is equivalent to $\Phi_i \leqslant l_i + \sum_{j\neq i} v_j s_j^*(v)$ for all $v \in V^n$. Thus, the most restrictive case gives $\Phi_i \leqslant l_i$. Finally, IIR requires

$$\mathbb{E}_{-i} v_i (s_i^*(v) - r_i) + \mathbb{E}_{-i} t_i^*(v_i) = \mathbb{E}_{-i} g(v) - \Phi_i - v_i r_i \geqslant 0 \quad \text{for all } v_i \in V,$$

or equivalently that $\Phi_i \leqslant C(r_i)$ with $C(r_i) := \inf_{v_i}\{\mathbb{E}_{-i} g(v) - v_i r_i\}$. Aggregating IIR and EPCC, *i.e.* $\Phi_i \leqslant l_i$ and $\Phi_i \leqslant C(r_i)$, respectively, gives $\Phi_i \leqslant \min\{C(r_i), l_i\}$ for all $i \in N$.

Take for instance, $\Phi_i = \min\{C(r_i), l_i\} - \frac{\sum_{k\in N}\min\{C(r_k),l_k\}-(n-1)G}{n}$. It is immediate that EABB is satisfied as $\sum_{i\in N} \Phi_i = (n-1)G$. Moreover, as condition (14) is satisfied, *i.e.* $\sum_i \min\{C(r_i), l_i\} \geqslant (n-1)G$, it is clear that $\Phi_i \leqslant \min\{C(r_i), l_i\}$ therefore satisfying IIR and EPCC as well. ■

Proof of Theorem 3. The dissolution mechanism $(s^*, \tilde{t})$ is EF and EPIC then, from Proposition 1.b., $\tilde{t}_i(v) = g(v) - v_i s_i^*(v) - k_i(v_{-i})$ for some function k_i independent of v_i. Let $K_i := \mathbb{E}_{-i} k_i(v_{-i})$. From EABB, $\mathbb{E}\sum_{i\in N} \tilde{t}_i(v) = (n-1)G - \sum_{i\in N} K_i = 0$. Take another dissolution mechanism (s^*, t) with

$$t_i(v) = g(v) - v_i s_i^*(v) - h_i(v) + \frac{n-1}{n} G - \phi_i - K_i,$$

where $h_i(v)$ is defined by equation (9) for which $\mathbb{E}_{-i} h_i(v_{-i}) = \frac{n-1}{n} G - \phi_i$ and $\sum_{i\in N} h_i(v) = (n-1)g(v) - \sum_{i\in N} \phi_i$. Then, it is immediate that (s^*, t) is EF and IIC. From (11), $\min_v g(v) - v_i s_i(v) - h_i(v) = -\frac{n-1}{n} G + \phi_i$ and then $\min_v t_i(v) = -K_i$. As $\tilde{t}$ is EPCC, $\tilde{t}_i(v) = g(v) - v_i s_i^*(v) - k_i(v_{-i}) \geqslant -l_i$ for all $i \in N$, $v \in V^n$ which implies that $\mathbb{E}_{-i}[g(v) - v_i s_i^*(v)] - K_i \geqslant -l_i$ for all $i \in N$, $v_i \in V$ and then that $-K_i \geqslant -l_i$ given that the infimum of $\mathbb{E}_{-i}[g(v) - v_i s_i^*(v)]$ is zero. It follows that $\min_v t_i(v) = -K_i \geqslant -l_i$ so that t is also EPCC. Straightforward computations shows that $\sum_{i\in N} t_i(v) = 0$ as $\sum_{i\in N} K_i = (n-1)G$. Finally, notice that $\mathbb{E}_{-i} t_i(v) = \mathbb{E}_{-i}[g(v) - v_i s_i^*(v)] - K_i = \mathbb{E}_{-i}\tilde{t}_i(v)$ implying that (s^*, t) is payoff equivalent to $(s^*, \tilde{t})$ at the interim stage and thus also IIR. ■

Proof of Theorem 4. The dissolution mechanism (s^*, t) is EF and IIC then, from Proposition 1.a., $t_i(v) = g(v) - v_i s_i^*(v) - h_i(v)$ where $\mathbb{E}_{-i} h_i(v) =: H_i$. Take another dissolution mechanism $(s^*, \tilde{t})$ where

$$\begin{aligned}\tilde{t}_i(v) &= t_i(v) + h_i(v) - H_i \\ &= g(v) - v_i s_i^*(v) - H_i.\end{aligned}$$

Then, from Proposition 1.b., $\tilde{t}$ is EF and EPIC. From EPCC, $t_i(v) = g(v) - v_i s_i^*(v) - h_i(v) \geqslant -l_i$ for all $i \in N$, $v \in V^n$ which implies that $\mathbb{E}_{-i}[g(v) - v_i s_i^*(v) - H_i \geqslant -l_i$ for all $i \in N$, $v_i \in V$. This is equivalent to $-H_i \geqslant -l_i$ for all $i \in N$. Then notice that $\min_v \tilde{t}_i(v) = -H_i \geqslant -l_i$ such that $\tilde{t}$ is also EPCC. Computing $\mathbb{E}\sum_{i\in N} \tilde{t}_i(v) = \mathbb{E}\sum_{i\in N} t_i(v) = 0$ and thus $\tilde{t}$ is EABB. Finally, as $\mathbb{E}_{-i}\tilde{t}_i(v) = \mathbb{E}_{-i} t_i(v)$ for all $i \in N$, $v_i \in V$, then $(s^*, \tilde{t})$ is interim payoff equivalent to (s^*, t) and also IIR. ■

Proof of Proposition 2. *(Only if)* Assume there exists an EF, IIC, IIR, EPBB and *EPCC* mechanism (s, t). From EPCC we have $t_i(v) \geqslant -l_i$ for all $i \in N$, $v \in V^n$. Then, simply let $\tilde{t}_i(v) = t_i(v)$ for all i, v and it is immediate that $\mathbb{E}_{-i}\tilde{t}_i(v_i, v_{-i}) \geqslant -l_i$ for all i,v. The mechanism $(s, \tilde{t})$ therefore satisfies all the properties of the mechanism (s, t) and also satisfies interim cash constraints.

(If) Assume there exists an EF, IIC, IIR, EPBB and *interim* cash-constrained mechanism $(s, \tilde{t})$. As $(s, \tilde{t})$ is IIC, the transfer rule is Groves in expectations and can be written as $\tilde{t}_i(v) = g(v) - v_i s_i(v) - h_i(v)$ with $\mathbb{E}_{-i} h_i(v) := H_i$ for all i, v_i. The interim cash constraints then writes $\mathbb{E}_{-i}\tilde{t}_i(v_i, v_{-i}) \geqslant -l_i$ for all $i \in N$, $v_i \in V$ or equivalently

$$G_i(v_i) - \mathbb{E}_{-i} v_i s_i(v) - H_i \geqslant -l_i \quad \text{for all} \quad i, v_i.$$

This inequality is equivalent to $H_i \leqslant \inf_{v_i}\{G_i(v_i) - \mathbb{E}_{-i}v_i s_i(v)\} + l_i$ for all $i \in N$. As the infimum is zero, we simply have $H_i \leqslant l_i$ for all $i \in N$. Remark that this is the same necessary condition to satisfy the cash constraints as in the proof of Theorem 1, *i.e.*, it is the same necessary condition than for a mechanism with EPCC. As the other conditions (EF, IIR and EPBB) are the same we can say that we have a mechanism $(s, \tilde{t})$ only if $\sum_i \min\{C_i, l_i\} \geqslant (n-1)G$. But from Theorem 1, we know that this condition is sufficient to construct a mechanism (s,t) satisfying EPCC. ■

Proof of Proposition 3. Starting with Proposition 3.a., assume that $\tilde{r}_i \leqslant \frac{1}{n}$ for all $i \in N$. Notice that $\max_{r\in\Delta^{n-1}} \sum_{i\in N} C(r_i) = \sum_{i\in N} C(\frac{1}{n})$ and thus for all $r \in \Delta^{n-1}$, $\sum_{i\in N} \min\{C(r_i), l_i\} \leqslant \sum_{i\in N} C(\frac{1}{n})$. It is then clear that choosing $r_i^* = \frac{1}{n}$ for all $i \in N$ is such that for each $i \in N$, $\min\{C(r_i^*), l_i\} = C(r_i^*) = C(\frac{1}{n})$ provided that $r_i^* \geqslant \tilde{r}_i$ for all $i \in N$. Hence, $\sum_{i\in N} \min\{C(r_i), l_i\} = \sum_{i\in N} C(\frac{1}{n})$ which is the upper bound.

Consider now Proposition 3.b., *i.e.* assume that $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$. Define $\mathcal{L}(r,\lambda) = \sum_{i\in N} \min\{C(r_i), l_i\} + \lambda(\sum_{i\in N} r_i - 1)$ where $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the constraint $\sum_{i\in N} r_i = 1$. Notice that $\sum_{i\in N} \min\{C(r_i), l_i\}$ is concave as $C(r_i)$ is concave for each $i \in N$ and differentiable everywhere except at $r_i = \tilde{r}_i$. Let $\delta_{r_i}\mathcal{L}(r,\lambda)$ denote the superdifferential of the Lagrangian in r_i, then

$$\delta_{r_i}\mathcal{L}(r,\lambda) = \lambda + \begin{cases} 0 & \text{if } r_i < \tilde{r}_i \\ [C'(\tilde{r}_i), 0] & \text{if } r_i = \tilde{r}_i \\ C'(r_i) & \text{if } r_i > \tilde{r}_i. \end{cases}$$

The necessary optimality condition writes $0 \in \delta_{r_i}\mathcal{L}(r,\lambda)$ for all $i \in N$. First, assume that there is at least one $r_j^* < \tilde{r}_j$. Then $\lambda = 0$ and $r_i > \tilde{r}_i$ is impossible as it is impossible to have $C'(r_i) = 0$ with $r_i > \tilde{r}_i$ (indeed $C'(r_i) = 0$ only occurs when $\underline{v} = 0$ and $r_i = 0$). But then, if all $r_i^* \leqslant \tilde{r}_i$ with one strict inequality at least, it follows that $\sum_{i\in N} r_i^* < \sum_{i\in N} \tilde{r}_i \leqslant 1$ which is also impossible. Therefore, it is necessary that $r_i \geqslant \tilde{r}_i$ for all $i \in N$. Assume now that $r_i > \tilde{r}_i$ for all $i \in N$. Then, the necessary optimality condition implies that $\lambda + C'(r_i^*) = 0$ for all $i \in N$. But then it follows that $r_i^* = \frac{1}{n}$ for all $i \in N$ which is impossible as some $\tilde{r}_i > \frac{1}{n}$ contradicting that $r_i^* > \tilde{r}_i$ for all $i \in N$.

Hence, the solution must be such $r_i^* \geqslant \tilde{r}_i$ for all $i \in N$ with at least one equality. Let $\mathcal{A} := \{i \in N \mid r_i^* > \tilde{r}_i\}$ and $\mathcal{B} := \{j \in N \mid r_j^* = \tilde{r}_j\}$. Then, for all $i \in \mathcal{A}$, $\lambda + C'(r_i^*) = 0$ implies that $\lambda > 0$ and $r_i^* = r_k^*$ for any two $i,k \in \mathcal{A}$. For any $i \in \mathcal{A}$, and let $r_i^* = \hat{r}$ with $\hat{r} := \frac{1-\sum_{j\in\mathcal{B}} \tilde{r}_j}{|\mathcal{A}|}$. As by assumption $\tilde{r}_1 \leqslant \cdots \leqslant \tilde{r}_n$ and for all $i \in \mathcal{A}$ it is necessary that $\hat{r} > \tilde{r}_i$, it is possible to rewrite $\mathcal{A} := \{i \in N \mid i < p\}$ and $\mathcal{B} := \{j \in N \mid j \geqslant p\}$ for some $p \in N \setminus \{1\}$ and $\hat{r} = \frac{1-\sum j\geqslant p \tilde{r}_j}{p-1}$. It is also necessary that $\hat{r} \leqslant \tilde{r}_j$ for all $j \in \mathcal{B}$. The solution therefore writes $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r}, \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ and $\max_{i<p} \tilde{r}_i < \hat{r} \leqslant \min_{j\geqslant p} \tilde{r}_j$. ■

APPENDIX B

Proof of Theorem 6. The right derivative of $\delta_i(v_i) \mid \cdot)$ writes:

$$\frac{\partial_+ \delta_i(v_i \mid \cdot)}{\partial \overline{m}_i} = \begin{cases} 0 & \text{if } v_i < \overline{m}_i \\ \frac{f(\overline{m}_i)}{1-F(\overline{m}_i)} \left[\frac{1}{1-F(\overline{m}_i)} \int_{\overline{m}_i}^{\overline{v}} \alpha(v_i \mid \lambda) dF(v_i) - \alpha(\overline{m}_i \mid \lambda) - \frac{\tau_i}{f(\overline{m}_i)} - \frac{\tau_i \overline{m}_i}{1-F(\overline{m}_i)} \right] & \text{if } v_i \geqslant \overline{m}_i, \end{cases}$$

then we simply have

$$\frac{\partial_+ \delta_i(v_i \mid \cdot)}{\partial \overline{m}_i} = \begin{cases} 0 & \text{if } v_i < \overline{m}_i \\ \frac{f(\overline{m}_i)}{1-F(\overline{m}_i)} \left[\delta_i(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) - \delta_i^-(\overline{m}_i \mid x_i, \overline{m}_i, \lambda) \right] & \text{if } v_i \geqslant \overline{m}_i. \end{cases}$$

Now, taking the right derivative of $\hat{\mathcal{L}}$ at $\overline{m}_i$ gives

$$\begin{aligned}\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i} &= \int_{V^n}\frac{\partial_+\max_{i\in N}\delta_i(\nu_i\mid\cdot)}{\partial\overline{m}_i}dF(\nu) - r_i\int_V\frac{\partial_+\delta_i(\nu_i\mid\cdot)}{\partial\overline{m}_i}dF(\nu_i)\\ &+ f(\overline{m}_i)\int_{V^{n-1}}\Big(\max\big\{\delta_i^-(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\} - \max\big\{\delta_i(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\}\Big)dF(\nu_{-i})\\ &- r_i\Big(\delta_i^-(\overline{m}_i\mid\cdot)-\delta_i(\overline{m}_i\mid\cdot)\Big)f(\overline{m}_i).\end{aligned}$$

Notice that the terms that depend on r_i cancel out. Thus we obtain

$$\begin{aligned}\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i} &= \int_{V^n}\frac{\partial_+\max_{i\in N}\delta_i(\nu_i\mid\cdot)}{\partial\overline{m}_i}dF(\nu)\\ &+ f(\overline{m}_i)\int_{V^{n-1}}\Big(\max\big\{\delta_i^-(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\} - \max\big\{\delta_i(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\}\Big)dF(\nu_{-i}).\end{aligned}$$

First, assume that $\delta_i^-(\overline{m}_i\mid\cdot) > \delta_i(\overline{m}_i\mid\cdot)$. This is equivalent to say that

$$\tau_i > \frac{\Big(\mathbb{E}\big[\alpha(\nu_i\mid\lambda)\mid\nu_i\geqslant\overline{m}_i\big]-\alpha(\overline{m}_i\mid\lambda)\Big)(1-F(\overline{m}_i))f(\overline{m}_i)}{1-F(\overline{m}_i)+\overline{m}_i f(\overline{m}_i)}.$$

Using the fact that $\partial_+\delta_j(\nu_j\mid\cdot)/\partial\overline{m}_i = 0$ for all $j\neq i$ and the expression of the right derivative of $\delta_i(\nu_i\mid\cdot)$, the first term of $\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i}$ rewrites

$$\begin{aligned}&\int_{V^n}\frac{\partial_+\max_{i\in N}\delta_i(\nu_i\mid\cdot)}{\partial\overline{m}_i}dF(\nu) =\\ &\int_{\nu\in V^n\,\mid\,\nu_i\in[\overline{m}_i,\overline{\nu}],\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i(\overline{m}_i\mid\cdot)}\frac{f(\overline{m}_i)}{1-F(\overline{m}_i)}\left(\delta_i(\overline{m}_i\mid\cdot)-\delta_i^-(\overline{m}_i\mid\cdot)\right)dF(\nu) =\\ &f(\overline{m}_i)\int_{\nu_{-i}\in V^{n-1}\,\mid\,\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i(\overline{m}_i\mid\cdot)}\left(\delta_i(\overline{m}_i\mid\cdot)-\delta_i^-(\overline{m}_i\mid\cdot)\right)dF_{-i}(\nu_{-i}).\end{aligned}$$

As for the second term, it rewrites

$$\begin{aligned}&f(\overline{m}_i)\int_{V^{n-1}}\Big(\max\big\{\delta_i^-(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\} - \max\big\{\delta_i(\overline{m}_i\mid\cdot),\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\big\}\Big)dF(\nu_{-i}) =\\ &f(\overline{m}_i)\int_{\nu_{-i}\in V^{n-1}\,\mid\,\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i(\overline{m}_i\mid\cdot)}\left(\delta_i^-(\overline{m}_i\mid\cdot)-\delta_i(\overline{m}_i\mid\cdot)\right)dF(\nu_{-i})\\ &+ f(\overline{m}_i)\int_{\nu_{-i}\in V^{n-1}\,\mid\,\delta_i(\overline{m}_i\mid\cdot)\leqslant\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i^-(\overline{m}_i\mid\cdot)}\left(\delta_i^-(\overline{m}_i\mid\cdot)-\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\right)dF(\nu_{-i}).\end{aligned}$$

As the first term of $\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i}$ cancels out with the first term of the above equation we finally get

$$\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i} = f(\overline{m}_i)\int_{\nu_{-i}\in V^{n-1}\,\mid\,\delta_i(\overline{m}_i\mid\cdot)\leqslant\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i^-(\overline{m}_i\mid\cdot)}\left(\delta_i^-(\overline{m}_i\mid\cdot)-\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\right)dF(\nu_{-i}).$$

Therefore, this expression is exactly the same as in Boulatov and Severinov (2018). It follows that is it not possible that $\delta_i^-(\overline{m}_i\mid\cdot) > \delta_i(\overline{m}_i\mid\cdot)$ and that the set $A_i(\cdot) := \{\nu_{-i}\in V^{n-1}\mid \delta_i(\overline{m}_i\mid\cdot)\leqslant\max_{j\neq i}\delta_j(\nu_j\mid\cdot)<\delta_i^-(\overline{m}_i\mid\cdot)\}$ has positive measure as it would imply that $\frac{\partial_+\hat{\mathcal{L}}}{\partial\overline{m}_i} > 0$, contradicting the optimality of $\overline{m}_i$.

The same reasoning as in Boulatov and Severinov (2018) can be done to prove that it is not possible to have $\delta_i^-(\overline{m}_i\mid\cdot) < \delta_i(\overline{m}_i\mid\cdot)$ and the set $B_i(\cdot) := \{\nu_{-i}\in V^{n-1}\mid \delta_i^-(\overline{m}_i\mid\cdot)<\max_{j\neq i}\delta_j(\nu_j\mid\cdot)\leqslant\delta_i(\overline{m}_i\mid\cdot)\}$ has a positive measure.

Let $\delta_i(\overline{m}_i\mid\cdot)$ be replaced by $\delta_i(\overline{m}_i)$ for convenience of the following lemmas.

Lemma 6 *Assume that for some* i *and* j, $\delta_i^-(\overline{m}_i)\leqslant\delta_j^-(\overline{m}_j)$. *Then* $\delta_i^-(\overline{m}_i)\leqslant\delta_i(\overline{m}_i)$.

Proof. Assume that $\delta_i^-(\overline{m}_i)\leqslant\delta_j^-(\overline{m}_j)$ for some i and j but $\delta_i(\overline{m}_i)<\delta_i^-(\overline{m}_i)$. Then, as for any $k\in N$, $\delta_k(\underline{\nu})=\underline{\nu}$ we must have $\delta_i^-(\overline{m}_i)>\underline{\nu}$. Then, for all $k\in N$ there exists a $\tilde{\nu}_k\in(0,\overline{m}_k]$ such that $\delta_k(\nu_k)<\delta_i^-(\overline{m}_i)$ for all $\nu_k\in[\underline{\nu},\tilde{\nu}_k)$.

At the same time, as $\underline{v} \leqslant \delta_i(\overline{m}_i) < \delta_i^-(\overline{m}_i) \leqslant \delta_j^-(\overline{m}_j)$ and $\delta_j(\underline{v}) = \underline{v}$, then, by continuity of $\delta_j(v_j)$ there must exist some $\tilde{v}_j$ such that $\delta_j(\tilde{v}_j) = \delta_i^-(\overline{m}_i)$ and a $b_j > 0$ (that can be made arbitrarily small) such that $\delta_i(\overline{m}_i) < \delta_j(v_j) < \delta_i^-(\overline{m}_i)$ for all $v_j \in (\tilde{v}_j - b_j, \tilde{v}_j)$. Hence, for all $v_j \in (\tilde{v}_j - b_j, \tilde{v}_j)$ and all $v_k \in [\underline{v}, \tilde{v}_k)$ for all $k \neq i, j$ we have $\delta_i(\overline{m}_i) < \max\{\delta_j(v_j), \max_{k\neq i,j} \delta_k(v_k)\} < \delta_i^-(\overline{m}_i)$. As both $(\tilde{v}_j - b_j, \tilde{v}_j)$ and $[\underline{v}, \tilde{v}_k)$ have non-zero measure, it follows that $A_i(\cdot)$ has positive measure which contradicts the optimality of $\overline{m}_i$. Thus, we must have $\delta_i^-(\overline{m}_i) \leqslant \delta_i(\overline{m}_i)$. ■

Lemma 7 *Let* $h_1 \in \arg\max_{j\in N} \delta_j^-(\overline{m}_j)$. *Assume either (i)* $\delta_i^-(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$ *or (ii)* $\delta_i^-(\overline{m}_i) = \delta_{h_1}^-(\overline{m}_{h_1})$ *and* $\delta_i(\overline{m}_i) \leqslant \delta_{h_1}(\overline{m}_{h_1})$. *Then* $\delta_i(\overline{m}_i) \leqslant \delta_i^-(\overline{m}_i)$.

Proof. Case (i). Assume (i) but suppose that $\delta_i^-(\overline{m}_i) < \delta_i(\overline{m}_i)$. Then, as $\delta_k(\underline{v}) = \underline{v}$ for all $k \in N$ we must have $\delta_i(\overline{m}_i) > \underline{v}$. For all $k \neq i, h_1$ is it clear that there exists $\tilde{v}_k \in (\underline{v}, \overline{m}_k]$ such that $\delta_k(v_k) < \delta_i(\overline{m}_i)$ for all $v_k \in [\underline{v}, \tilde{v}_k)$.

Additionally either $\underline{v} \leqslant \delta_i^-(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1}) \leqslant \delta_i(\overline{m}_i)$ or $\underline{v} \leqslant \delta_i^-(\overline{m}_i) < \delta_i(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$. In both cases, there must exist a $\tilde{m}_{h_1} \in (\underline{v}, \overline{m}_{h1}]$ such that $\delta_{h_1}(v_{h_1}) < \delta_i(\overline{m}_i)$ for all $v_{h1} \in [\underline{v}, \tilde{m}_{h_1})$. As $\delta_i^-(\overline{m}_i) < \delta_i(\overline{m}_i)$ there must be a $b_{h_1} > 0$ (that can be made arbitrarily small) such that $\delta_i^-(\overline{m}_i) < \delta_{h_1}(v_{h_1}) < \delta_i(\overline{m}_i)$ for all $v_{h_1} \in (\tilde{m}_{h_1} - b_{h_1}, \tilde{m}_{h_1})$.

It immediately follows that for all $k \neq i, h_1$, $v_k \in [\underline{v}, \tilde{v}_k)$ and for all $v_{h_1} \in (\tilde{m}_{h_1} - b_{h_1}, \tilde{m}_{h_1})$, $\delta_i(\overline{m}_i) < \max\{\delta_{h_1}^-(v_{h_1}), \max_{k\neq i,h_1} \delta_k(v_k)\} < \delta_i(\overline{m}_i)$, *i.e.*, the set $B_i(\cdot)$ has a positive measure, which contradicts the optimality of $\overline{m}_i$.

Case (ii). Assume now (ii) and $\delta_i^-(\overline{m}_i) < \delta_i(\overline{m}_i)$. This would imply $\delta_i^-(\overline{m}_i) = \delta_{h_1}^-(\overline{m}_{h_1}) < \delta_i(\overline{m}_i) \leqslant \delta_{h_1}(\overline{m}_{h_1})$. It follows that $\delta_i(\overline{m}_i) > \underline{v}$ and also that the set $\{v_k \mid \delta_k(v_k) < \delta_{h_1}(\overline{m}_{h_1})\}$ includes $[\underline{v}, \overline{m}_k)$ for all $k \neq i, h_1$ as by definition $\delta_i^-(\overline{m}_i) = \max_{j\in N} \delta_j^-(\overline{m}_j) < \delta_i(\overline{m}_i) \leqslant \delta_{h_1}(\overline{m}_{h_1})$.

Additionally, we have the $\{v_i \mid \delta_{h_1}^-(\overline{m}_{h_1}) < \delta_i(v_i) \leqslant \delta_{h_1}(\overline{m}_{h_1})\} = [\overline{m}_i, \overline{v}]$ which has positive measure as $\overline{m}_i < \overline{v}$ (given that we have a jump in $\delta_i(\cdot)$).

Therefore, for all $v_i \in [\overline{m}_i, \overline{v}]$ and $v_k \in [\underline{v}, \overline{m}_k)$ for $k \neq i, h_1$, we have that $\delta_{h_1}^-(\overline{m}_{h_1}) < \max\{\delta_i(v_i), \max_{k\neq i,h_1} \delta_k(v_k)\} \leqslant \delta_{h_1}(\overline{m}_{h_1})$ and thus the set $B_{h1}(\cdot)$ has positive measure which contradicts the optimality of $\overline{m}_{h_1}$. ■

Lemma 8 *Let* $h_1 \in \arg\max_{j\in N} \delta_j^-(\overline{m}_j)$ *and assume that for* $i \neq h_1$, $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$ *then we must have* $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i) < \delta_{h_1}(\overline{m}_{h_1})$.

Proof. Assume instead that $\delta_{h_1}(\overline{m}_{h_1}) \leqslant \delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$. It follows that $\delta_{h_1}^-(\overline{m}_{h_1}) > \underline{v}$. Hence, as for all $k \in N$, $\delta_k(\underline{v}) = \underline{v}$, there exists a $\tilde{v}_k \in (\underline{v}, \overline{m}_k]$ such that $\delta_k(v_k) < \delta_{h_1}^-(\overline{m}_{h_1})$ for all $v_k \in [\underline{v}, \tilde{v}_k)$. Then, as $\underline{v} \leqslant \delta_{h_1}(\overline{m}_{h_1}) \leqslant \delta_i(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$, there must exists a $\tilde{m}_i \in (\underline{v}, \overline{m}_i]$ and a $b_i > 0$ such that $\delta_{h_1}(\overline{m}_{h_1}) \leqslant \delta_i(v_i) < \delta_{h_1}^-(\overline{m}_{h_1})$ for all $v_i \in (\tilde{m}_i - b_i, \tilde{m}_i)$. Then, the set A_{h_1} would have positive measure and this would contradict the optimality of $\overline{m}_{h_1}$.

Lemma 9 *There exists a unique* $z_1 \in \arg\max_{j\in N} \delta_j^-(\overline{m}_j)$ *and* $z_1 \in \arg\max_{j\in H_1} \delta_j(\overline{m}_j)$ *such that for all* $i \neq z_1$ *we have*

$$\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i).$$

and either

$$\min\{\delta_{z_1}^-(\overline{m}_{h_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \max_{i\neq z_1} \delta_i(\overline{m}_i) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$$

or

$$\delta_{z_1}(\overline{m}_{z_1}) \geqslant \delta_{z_1}^-(\overline{m}_{z_1}) = \max_{i\neq z_1} \delta_i(\overline{m}_i) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i).$$

Proof. Let $H_1 = \{h_1 \in N \mid h_1 = \arg\max_{j\in N} \delta_j^-(\overline{m}_j)\}$, *i.e.*, the set of agents for which $\delta_j^-(\overline{m}_j)$ is the maximum. Let $Z_1 = \{z_1 \in H_1 \mid z_1 = \arg\max_{j\in N} \delta_j(\overline{m}_j)\}$, *i.e.*, the set of agents in H_1 for which $\delta_j(\overline{m}_j)$ is the maximum.

Then, for all $i \notin H_1$ we must have $\delta_i^-(\overline{m}_i) < \delta_{h_1}^-(\overline{m}_{h_1})$ and both Lemma 6 and Lemma 7 (through condition (i)) apply. Hence $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i)$ for all $i \notin H_1$.

Now consider $i \in H_1$.

1. Case $|H_1| > 1$. Clearly, for all $i \in H_1$, Lemma 6 applies and thus $\delta_i^-(\overline{m}_i) \leqslant \delta_i(\overline{m}_i)$ for all $i \in H_1$.
   (a) Case $|Z_1| > 1$ then all $i \in H_1 \cap \overline{Z}_1$ have $\delta_i^-(\overline{m}_i) < \delta_j(\overline{m}_j)$ for $j \in Z_1$ and all $k, j \in Z_1$ have $\delta_k^-(\overline{m}_k) < \delta_j(\overline{m}_j)$. Thus Lemma 7 (part (ii)) applies for all $i \in H_1 \cap \overline{Z}_1$ and all $i \in Z_1$. Hence $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i)$ for all $i \in N$.
   (b) If $|Z_1| = 1$, then Lemma 7 (part (ii)) applies to all $i \in H_1 \cap \overline{Z}_1$ but not to $z_1 \in Z_1 = \{z_1\}$.
2. Case $|H_1| = 1$. Then it is immediate that $|Z_1| = 1$ as well. Then Lemma 7 (part (ii)) applies to all $i \in H_1 \cap \overline{Z}_1$ but not to $z_1 \in Z_1 = \{z_1\}$.

Then either $|Z_1| > 1$ and $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i)$ for all $i \in N$ or $|Z_1| = 1$ and $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i)$ for all $i \neq z_1$. By definition of Z_1, when $|Z_1| = 1$, $\delta_{z_1}(\overline{m}_{z_1}) > \max_{i\neq z_1} \delta_i(\overline{m}_i) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$. As $z_1 \in H_1$ we also have that $\delta_{z_1}^-(\overline{m}_{z_1}) \geqslant \max_{i\neq z_1} \delta_i(\overline{m}_i) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$. Therefore, we must have $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} \geqslant \max_{i\neq z_1} \delta_i^-(\overline{m}_i) = \max_{i\neq z_1} \delta_i(\overline{m}_i)$.

The only reason we could have an equality in the above inequality is when $\delta_{z_1}^-(\overline{m}_{z_1}) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$ as we have shown that $\delta_{z_1}(\overline{m}_{z_1}) > \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$. But if we assume $\delta_{z_1}^-(\overline{m}_{z_1}) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i)$ then it means that Lemma 6 applies and thus we must have $\delta_{z_1}^-(\overline{m}_{z_1}) \leqslant \delta_{z_1}(\overline{m}_{z_1})$.

Hence either we have $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \max_{i\neq z_1} \delta_i^-(\overline{m}_i) = \max_{i\neq z_1} \delta_i(\overline{m}_i)$ or we have $\max_{i\neq z_1} \delta_i(\overline{m}_i) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i) = \delta_{z_1}^-(\overline{m}_{z_1}) \leqslant \delta_{z_1}(\overline{m}_{z_1})$. ■

Lemma 10 *There exists $\overline{m}_{z_1} > \overline{m}_j$ for all $j \neq z_1$ if and only if either for all $j \neq z_1$, $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \delta_j^-(\overline{m}_j) = \delta_j(\overline{m}_j)$ or $\delta_{z_1}(\overline{m}_{z_1}) > \delta_{z_1}^-(\overline{m}_{z_1}) = \max_{i\neq z_1} \delta_i^-(\overline{m}_i) = \max_{i\neq z_1} \delta_i(\overline{m}_i)$.*

Proof. *(Only if)* Assume $\overline{m}_p > \overline{m}_j$ for all $j \neq p$. First, let us show that this implies that $p \in \arg\max_{j\in N} \delta_j^-(\overline{m}_j)$. Suppose the contrary, that is, $p \notin H_1$. Then by Lemma 6 and 7 we must have $\delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p)$ and thus $\delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p) < \delta_i^-(\overline{m}_i)$ for some $i \in H_1$. From Lemma 8 it must then also be the case that $\delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p) < \delta_i(\overline{m}_i)$ for $i \in H_1$. But then $\min\{\delta_i^-(\overline{m}_i), \delta_i(\overline{m}_i)\} > \delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p)$. For any j, let $\Delta_j(\overline{m}_j) := \frac{\int_{\overline{m}_j}^{\overline{v}} \alpha(v_j|\lambda)dF(v_j) + \overline{m}_j f(\overline{m}_j)\alpha(\overline{m}_j|\lambda)}{1-F(\overline{m}_j)+\overline{m}_j f(\overline{m}_j)}$, *i.e.* the value of $\delta_i(\overline{m}_i)$ when it is continuous at $\overline{m}_i$. Notice that when $\min\{\delta_i^-(\overline{m}_i), \delta_i(\overline{m}_i)\} = \delta_i^-(\overline{m}_i)$ then we must have $\tau_i \leqslant \hat{\tau}_i$ and then it follows that $\delta_i^-(\overline{m}_i) \leqslant \Delta_i(\overline{m}_i)$. Hence we must have $\Delta_i(\overline{m}_i) \geqslant \delta_i^-(\overline{m}_i) > \Delta_p(\overline{m}_p) = \delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p)$. However, from Lemma **??**, $\Delta_j(\overline{m}_j)$ is increasing in $\overline{m}_j$ and thus this would imply that $\overline{m}_i > \overline{m}_p$, a contradiction. Now when $\min\{\delta_i^-(\overline{m}_i), \delta_i(\overline{m}_i)\} = \delta_i(\overline{m}_i)$ we must have $\tau_i \geqslant \hat{\tau}_i$ and thus $\delta_i(\overline{m}_i) \leqslant \Delta_i(\overline{m}_i)$. This would then imply $\Delta_i(\overline{m}_i) > \Delta_p(\overline{m}_p)$ and thus $\overline{m}_i > \overline{m}_p$, again a contradiction. Hence, if $\overline{m}_p > \overline{m}_j$, we must have $p \in H_1$.

From now on, we know that $p \in H_1$, that is, $\delta_p^-(\overline{m}_p) \geqslant \delta_j^-(\overline{m}_j)$ for all $j \in N$. Let us consider the following subcases.

1. Assume $p \notin Z_1$.
   (a) If $|Z_1| = 1$ then from Lemma 9, there is a unique $i \in Z_1$ such that $\min\{\delta_i^-(\overline{m}_i), \delta_i(\overline{m}_i)\} \geqslant \delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p)$. But then, if $\delta_i(\overline{m}_i) \geqslant \delta_i^-(\overline{m}_i)$ we must have $\tau_i \leqslant \hat{\tau}_i$, $\Delta_i(\overline{m}_i) \geqslant \delta_i^-(\overline{m}_i) \geqslant \delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p) = \Delta_p(\overline{m}_p)$ and thus $\overline{m}_i \geqslant \overline{m}_p$ which is a contradiction. If otherwise $\delta_i(\overline{m}_i) \leqslant \delta_i^-(\overline{m}_i)$ then we must have $\Delta_i(\overline{m}_i) \geqslant \delta_i(\overline{m}_i) \geqslant \delta_p^-(\overline{m}_p) = \delta_p(\overline{m}_p) = \Delta_p(\overline{m}_p)$. Which also leads to the contradiction that $\overline{m}_i \geqslant \overline{m}_p$. Then, if $p \notin Z_1$ we cannot have $|Z_1| = 1$.
   (b) If $|Z_1| > 1$ then we have $\delta_j^- = \delta_j$ for all $j \in N$. As $p \in H_1$ and $|Z_1| > 1$ we must have $|H_1| > 1$ and then there exists a $i \in H_1$, $i \neq p$ such that $\delta_p^- = \delta_p = \delta_i^- = \delta_i$. It directly follows that $\Delta_p = \Delta_i$ and thus $\overline{m}_p = \overline{m}_i$, which is a contradiction. Therefore, we must have $p \in Z_1$.

2. Assume now that $p \in Z_1$.

   (a) If $|H_1| = 1$ then $\delta_p^- > \delta_j^-$ for all $j \neq p$. We also have $|Z_1| = 1$ and thus $\delta_p > \delta_j$ for all $j \neq p$. As $|H_1| = 1$, then $H_1 = \{p\}$ and thus for all $j \neq p$, $\delta_j^- = \delta_j$. Hence $\min\{\delta_p^-, \delta_p\} > \delta_j^- = \delta_j$ for all $j \neq p$, which is the first condition of the Lemma.

   (b) If $|H_1| > 1$ and $|Z_1| = 1$. Then $\delta_p > \delta_j$ for all $j \neq p$. As $p \in H_1$ and $|H_1| > 1$, then from Lemma 6 we must have $\delta_p^- \leqslant \delta_p$. But as $|H_1| > 1$, there must exist a $i \in H_1$, $i \neq p$ such that $\delta_i^- = \delta_p^-$. As $i \in H_1$ but $i \notin Z_1$ then $\delta_i^- = \delta_i$. It follows that we must have $\delta_p > \delta_p^- = \delta_i^- = \delta_i$ which is the second condition of the Lemma.

   (c) If $|H_1| > 1$ and $|Z_1| > 1$. Then we must have $\delta_j^- = \delta_j$ for all $j \in N$. Then take any $i \in Z_1$, $i \neq p$ which exists as $|Z_1| > 1$, it follows that $\delta_p = \delta_p^- = \delta_i = \delta_i^-$. However, this implies that $\Delta_p = \Delta_i$ and thus that $\overline{m}_p = \overline{m}_i$ which is a contradiction.

Then, assuming $\overline{m}_p > \overline{m}_i$ implies that for any $j \neq p$, either $\min\{\delta_p^-, \delta_p\} > \delta_j^- = \delta_j$ or $\delta_p > \delta_p^- = \delta_j^- = \delta_j$ which concludes the proof of the *only if* statement.

(If) For any $j \neq p$ we have $\delta_j^- = \delta_j = \Delta_j$. Assume first that $\min\{\delta_p^-, \delta_p\} > \delta_j^- = \delta_j$ for some $j \neq p$. If $\delta_p \geqslant \delta_p^-$ then we must have $\tau_p \leqslant \hat{\tau}_p$ and thus $\Delta_p \geqslant \delta_p^- > \delta_j^- = \delta_j = \Delta_j$. This implies that $\overline{m}_p > \overline{m}_j$. Same logic applies when $\delta_p \leqslant \delta_p^-$ as this implies that $\tau_p \geqslant \hat{\tau}_p$, $\Delta_p \geqslant \delta_p > \delta_j^- = \delta_j = \Delta_j$ and thus $\overline{m}_p > \overline{m}_j$. Assume now that for some $j \neq p$, $\delta_p > \delta_p^- = \delta_j^- = \delta_j$. Then, $\tau_p < \hat{\tau}_p$ and $\Delta_p > \delta_p^- = \delta_j^- = \delta_j = \Delta_j$, implying $\overline{m}_p > \overline{m}_j$. ■

Lemma 11 *Let* $z_1 \in \arg\max_{j \in N} \delta_j^-(\overline{m}_j)$*, for any* $j \neq z_1$ *it is not possible that* $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \delta_j^-(\overline{m}_j) = \delta_j(\overline{m}_j)$*. Therefore, only the case* $\delta_{z_1}(\overline{m}_{z_1}) > \delta_{z_1}^-(\overline{m}_{z_1}) = \max_{i \neq z_1} \delta_i^-(\overline{m}_i) = \max_{i \neq z_1} \delta_i(\overline{m}_i)$ *is possible.*

Proof. Assume that there is an $\overline{m}_{z_1}$ such that $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \delta_j^-(\overline{m}_j) = \delta_j(\overline{m}_j)$ for all $j \neq z_1$. Notice that for any $k \in N$, $\delta_k(\underline{v}) = \underline{v}$ and $\min\{\delta_{z_1}^-(\overline{m}_{z_1}), \delta_{z_1}(\overline{m}_{z_1})\} > \overline{v}$. So, there must exist a $\tilde{m}_{z_1} < \overline{m}_{z_1}$ such that for all $v_{z_1} \in [\tilde{m}_{z_1}, \overline{m}_{z_1})$ we have $\max_{i \neq z_1} \delta_i(\overline{m}_i) < \delta_{z_1}(v_{z_1}) \leqslant \delta_{z_1}(\overline{m}_{z_1})$. But then, for all those $v_{z_1} \in [\tilde{m}_{z_1}, \overline{m}_{z_1})$, we have $\delta_{z_1}(v_{z_1}) > \max_{i \neq z_1} \max_{v_i \in V} \delta_i(v_i)$ and thus $S_{z_1}(v_{z_1}) = 1$ for all $v_{z_1} \in [\tilde{m}_{z_1}, \overline{m}_{z_1})$. It follows that $T_{z_1}(v_{z_1}) = T_{z_1}(\overline{v})$. But then, this means that $\overline{m}_{z_1}$ is not the smallest value in V before transfers become constant due to cash-constraints. Therefore, their must exist a $\hat{m}_{z_1}$ such that $\delta_{z_1}^-(\hat{m}_{z_1}) = \max_{i \neq z_1} \delta_i(\overline{m}_i)$. ■

Proof of Theorem 7. Assume that for a given $x^* = (x_1^*, \ldots, x_n^*)$, the vector $\mathcal{O}^* := (s^*, U^*, \overline{m}^*, \tau^*, \chi^*)$ solves (A) and satisfies (B). Then, from (A) we must have that $s_i^*(v)$ writes

$$s_i^*(v_i, v_{-i}) = \begin{cases} 1 & \text{if } \delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda) > \max_{j \neq i} \delta_j(v_j \mid x_j^*, \overline{m}_j^*, \lambda) \\ \text{something in } [0,1] & \text{if } \delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda) = \max_{j \neq i} \delta_j(v_j \mid x_j^*, \overline{m}_j^*, \lambda) \\ 0 & \text{if } \delta_i(v_i \mid x_i, \overline{m}_i^*, \lambda) < \max_{j \neq i} \delta_j(v_j \mid x_j^*, \overline{m}_j^*, \lambda) \end{cases}$$

As shown previously, $\overline{m}^*$ is such that for all $i \in N$, $\delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda)$ is nondecreasing w.r.t. v_i. It follows that $S_i(v_i)$ is also nondecreasing in v_i. From (B), $V^*(S_i^*) = [x_i^*, y_i^*]$ for all $i \in N$ and thus s_i^* satisfies all constraints of the original problem.

Notice also that for any $v_i^* \in V^*(S_i^*) = [x_i^*, y_i^*]$

$$\begin{aligned} \mathbb{E}\Big[\sum_{i \in N} \big(S_i^*(v_i) - r_i\big)\delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda)\Big] - \sum_{i \in N} (\lambda - \chi_i^* - \tau_i^*) U_i(x_i^*) + \sum_{i \in N} \tau_i^* l_i = \\ \mathbb{E}\Big[\sum_{i \in N} \big(S_i^*(v_i) - r_i\big)\Gamma_i(v_i \mid v_i^*, \overline{m}_i^*, \lambda)\Big] - \sum_{i \in N} (\lambda - \chi_i^* - \tau_i^*) U_i(v_i^*) + \sum_{i \in N} \tau_i^* l_i. \end{aligned} \tag{31}$$

This equality stems from two facts.

(i) We have that $\delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda) = \Gamma_i(v_i \mid v_i^*, \overline{m}_i^*, \lambda)$ for all $v_i \notin [x_i^*, y_i^*]$ and $v_i^* \in [x_i^*, y_i^*]$ by definition of $\delta_i(\cdot)$ so that the expectation is the same on both sides for $v_i \notin [x_i^*, y_i^*]$. For $v_i \in [x_i^*, y_i^*]$, δ_i and Γ_i differ, but at the same time we have that $S_i^*(v_i) = r_i$ as s^* satisfies (B). Thus the expectation is the same on both sides.

(ii) As $V^*(S_i^*) = [x_i^*, y_i^*]$ it is clear that $U_i(v_i^*) = U_i(x_i^*)$.

Now, by definition of $\mathcal{O}^*$, we must have that for all $\hat{\mathcal{O}} := (\hat{s}, \hat{U}, \hat{m}, \hat{\tau}, \hat{\chi})$

$$\mathbb{E}\big[\sum_{i\in N}\big(S_i^*(v_i)-r_i\big)\delta_i(v_i \mid x_i^*, \overline{m}_i^*, \lambda)\big] - \sum_{i\in N}(\lambda-\chi_i^*-\tau_i^*)U_i(x_i^*) + \sum_{i\in N}\tau_i^* l_i \geqslant \mathbb{E}\big[\sum_{i\in N}\big(\hat{S}_i(v_i)-r_i\big)\delta_i(v_i \mid x_i^*, \hat{m}_i, \lambda)\big] - \sum_{i\in N}(\lambda-\hat{\chi}_i-\hat{\tau}_i)\hat{U}_i + \sum_{i\in N}\hat{\tau}_i l_i. \tag{32}$$

At the same time we also have that for any $\hat{v}_i \in V^*(\hat{S}_i)$

$$\mathbb{E}\big[\sum_{i\in N}\big(\hat{S}_i(v_i)-r_i\big)\delta_i(v_i \mid x_i^*, \hat{m}_i, \lambda)\big] - \sum_{i\in N}(\lambda-\hat{\chi}_i-\hat{\tau}_i)\hat{U}_i + \sum_{i\in N}\hat{\tau}_i l_i \geqslant \mathbb{E}\big[\sum_{i\in N}\big(\hat{S}_i(v_i)-r_i\big)\Gamma_i(v_i \mid \hat{v}_i, \hat{m}_i, \lambda)\big] - \sum_{i\in N}(\lambda-\hat{\chi}_i-\hat{\tau}_i)\hat{U}_i + \sum_{i\in N}\hat{\tau}_i l_i, \tag{33}$$

as for all $v_i < \hat{v}_i$ we have $\delta_i(v_i \mid x_i^*, \hat{m}_i, \lambda) \leqslant \Gamma_i(v_i \mid \hat{v}_i, \hat{m}_i, \lambda)$ and $\hat{S}_i(v_i) - r_i \leqslant 0$ and for all $v_i > \hat{v}_i$ we have $\delta_i(v_i \mid x_i^*, \hat{m}_i, \lambda) \geqslant \Gamma_i(v_i \mid \hat{v}_i, \hat{m}_i, \lambda)$ and $\hat{S}_i(v_i) - r_i \geqslant 0$. [24] Hence, for all $v_i \in V$ we have $\big(\hat{S}_i(v_i) - r_i\big)\delta_i(v_i \mid x_i^*, \hat{m}_i, \lambda) \geqslant \big(\hat{S}_i(v_i) - r_i\big)\Gamma_i(v_i \mid \hat{v}_i, \hat{m}_i, \lambda)$.

But then combining equation (31), (32) and (33) we get that for all $\hat{\mathcal{O}}$ and $\hat{v}_i \in V^*(\hat{S}_i)$

$$\mathbb{E}\big[\sum_{i\in N}\big(S_i^*(v_i)-r_i\big)\Gamma_i(v_i \mid v_i^*, \overline{m}_i^*, \lambda)\big] - \sum_{i\in N}(\lambda-\chi_i^*-\tau_i^*)U_i(v_i^*) + \sum_{i\in N}\tau_i^* l_i \geqslant \mathbb{E}\big[\sum_{i\in N}\big(\hat{S}_i(v_i)-r_i\big)\Gamma_i(v_i \mid \hat{v}_i, \hat{m}_i, \lambda)\big] - \sum_{i\in N}(\lambda-\hat{\chi}_i-\hat{\tau}_i)\hat{U}_i + \sum_{i\in N}\hat{\tau}_i l_i, \tag{34}$$

which directly means that $\mathcal{O}^*$ is also the maximum of the original problem. ■

Proof of Corollary 1. Assume first that τ_i is defined by equation (24). It can easily be rewritten as

$$\hat{\tau}_i = \frac{f(\overline{m}_i)\int_{\overline{m}_i}^{\overline{v}}\alpha(v_i \mid \lambda)dF(v_i) - \alpha(\overline{m}_i \mid \lambda)(1-F(\overline{m}_i))f(\overline{m}_i)}{1-F(\overline{m}_i)+\overline{m}_i f(\overline{m}_i)}.$$

Differentiating w.r.t. to $\overline{m}_i$ gives the following condition for the numerator (after factorization by $(1 - F(\overline{m}_i))$ which is positive)

$$\Big(f'(\overline{m}_i)\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \alpha'(\overline{m}_i \mid \lambda))f(\overline{m}_i) - \alpha(\overline{m}_i \mid \lambda)f'(\overline{m}_i)\Big)\Big(1-F(\overline{m}_i)+\overline{m}_i f(\overline{m}_i)\Big) - \overline{m}_i f'(\overline{m}_i)\Big(\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \alpha(\overline{m}_i \mid \lambda)\Big)f(\overline{m}_i) \leqslant 0,$$

which reduces to

$$-\alpha'(\overline{m}_i \mid \lambda))f(\overline{m}_i)\Big(1-F(\overline{m}_i)+\overline{m}_i f(\overline{m}_i)\Big) + f'(\overline{m}_i)(1-F(\overline{m}_i))\Big[\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \alpha(\overline{m}_i \mid \lambda)\Big] \leqslant 0.$$

The above inequality holds as α' is positive, f is nonincreasing and $\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \alpha(\overline{m}_i \mid \lambda) \geqslant 0$. Then, τ_i is decreasing in $\overline{m}_i$.

Now assume instead that τ_{z_1} is defined by equation (25). For a given $\overline{m}_{-z_1}$ the right-hand side of (25) is constant. Fix τ_{z_1} and assume $\overline{m}_{z_1}$ increases. Then $\alpha(\overline{m}_{z_1} \mid \lambda) + \frac{\tau_{z_1}}{f(\overline{m}_{z_1})}$ increases as $\alpha(v_i \mid \lambda)$ is increasing in v_i and f is nonincreasing. Hence, to maintain the equality defined by equation (25) it is necessary that τ_{z_1} decreases.■

[24] $\hat{S}_i(v_i) - r_i \leqslant 0$ ($\geqslant 0$) for all $v_i < \hat{v}_i$ ($v_i > \hat{v}_i$) directly stems from the fact that $\hat{v}_i \in V^*(\hat{S}_i)$.

Proof of Corollary 2. Assume that for $i \in N$, $\delta_i^-(\overline{m}_i) = \delta_i(\overline{m}_i)$. This implies that τ_i is defined by equation (24). Therefore,

$$\begin{aligned}
\delta_i(\overline{m}_i) &= \mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \frac{\tau_i \overline{m}_i}{1 - F(\overline{m}_i)} \\
&= \mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \overline{m}_i \frac{\Big(\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] - \alpha(\overline{m}_i \mid \lambda)\Big) f(\overline{m}_i)}{1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)} \\
&= \frac{(1 - F(\overline{m}_i))\mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] + \overline{m}_i f(\overline{m}_i)\alpha(\overline{m}_i \mid \lambda)}{1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)} \\
&= \frac{\int_{\overline{m}_i}^{\overline{v}} \alpha(v_i \mid \lambda) dF(v_i) + \overline{m}_i f(\overline{m}_i)\alpha(\overline{m}_i \mid \lambda)}{1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)}.
\end{aligned}$$

Differentiating this expression w.r.t. $\overline{m}_i$ gives the following numerator

$$\begin{aligned}
&\overline{m}_i f(\overline{m}_i)\alpha'(\overline{m}_i \mid \lambda)][1 - F(\overline{m}_i) + \overline{m}_i f(\overline{m}_i)] + \\
&\quad (1 - F(\overline{m}_i))\overline{m}_i f'(\overline{m}_i)\Big[\alpha(\overline{m}_i \mid \lambda) - \frac{1}{1 - F(\overline{m}_i)} \int_{\overline{m}_i}^{\overline{v}} \alpha(v_i \mid \lambda) dF(v_i)\Big].
\end{aligned} \tag{35}$$

Given the assumption that f is nonincreasing, that $\alpha(\cdot)$ is increasing and that $\alpha(\overline{m}_i \mid \lambda) - \frac{1}{1-F(\overline{m}_i)} \int_{\overline{m}_i}^{\overline{v}} \alpha(v_i \mid \lambda) dF(v_i) = \alpha(\overline{m}_i \mid \lambda) - \mathbb{E}\big[\alpha(v_i \mid \lambda) \mid v_i \geqslant \overline{m}_i\big] \leqslant 0$ then (35) is positive and thus $\delta_i(\overline{m}_i)$ is increasing in $\overline{m}_i$ whenever $\delta_i(\overline{m}_i)$ is continuous at $\overline{m}_i$. ■

Proof of Corollary 3. (i) Take any $\overline{m} \in \times_{i \in N}(y_i, \overline{v}]$. Then for all $\overline{m}_i \leqslant \overline{m}_j$ for some j, τ_i is uniquely defined by (24). If it exists, $\overline{m}_{z_1} > \max_{i \neq z_1} \overline{m}_i$ defines τ_{z_1} from equation (25).

(ii) Now take any τ. Notice first that for all $i \in N$, $\overline{m}_i$ is decreasing in τ_i. Indeed, for any $\overline{m}_i$ satisfying (24), the RHS is decreasing in $\overline{m}_i$. Now for $\overline{m}_{z_1}$ satisfying (25), it is clear that the RHS does not depend on τ_{z_1} nor $\overline{m}_{z_1}$. Then, if τ_{z_1} increases, the only way to satisfy the equality is that $\overline{m}_{z_1}$ decreases as $\alpha(v_{z_1} \mid \lambda) + \frac{\tau_{z_1}}{f(\overline{v}_{z_1})}$ is increasing in v_{z_1}.

Then, for any i, j such that $\tau_i \geqslant \tau_j$ we must have $\overline{m}_i \leqslant \overline{m}_j$. It follows that $\overline{m}_i$ is uniquely defined by (24) as the RHS is decreasing in $\overline{m}_i$. Finally, if there exists a $\tau_{z_1} < \min_{i \neq z_1} \tau_j$ then we must have $\max_{i \neq z_1} \overline{m}_i < \overline{m}_{z_1}$ and $\overline{m}_{z_1}$ solves (25). The RHS of (25) depends only on $\min_{i \neq z_1} \tau_i$ and thus there exists a unique $\overline{m}_{z_1}$ that solves (25) for a given τ_{z_1} given that $\alpha(v_{z_1} \mid \lambda) + \frac{\tau_{z_1}}{f(v_{z_1})}$ is increasing in v_{z_1}. ■