Game Theory: Dynamic Games of Incomplete Information

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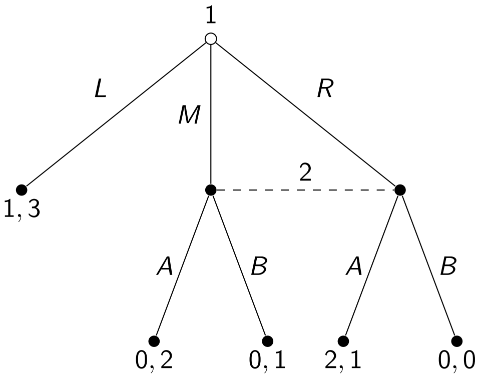
Introducing example

Consider the following game inspired by Selten (1975):

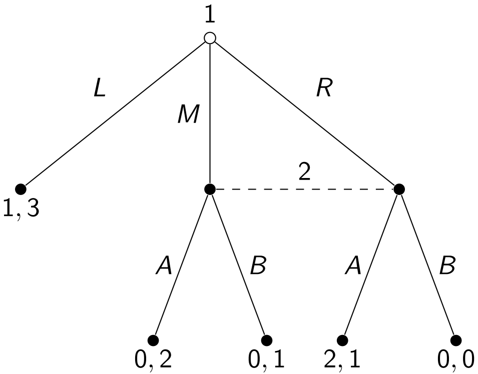


Player 2 observes whether player 1 played *L* or not.

- ▷ But cannot distinguish *M* from *R*

Introducing example

It can be seen as a dynamic game of **imperfect** information.

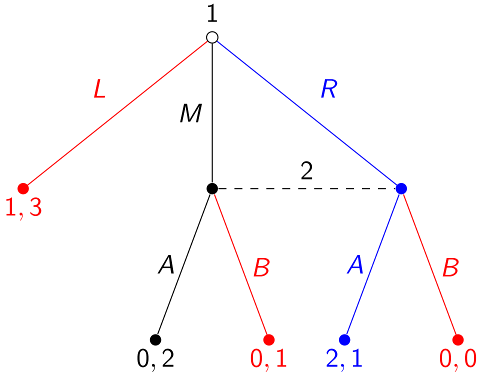


1\2	A	B
L	1,3	1,3
M	0,2	0,1
R	2,1	0,0

Introducing example

There are **two pure-strategy Nash Equilibria** in this game:

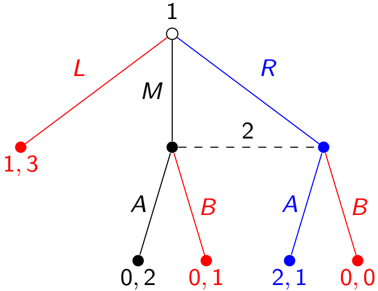
- ▷ (L, B) and (R, A)



1\2	A	B
L	1,3	1,3
M	0,2	0,1
R	2,1	0,0

Introducing example

Do you find the pure-strat. NE (L, B) *satisfactory?*



In (L, B), P2 *threatens* P1 to play B if the latter chooses M or R.

- ▷ B is a **non-credible threat**
- ▷ If P1 plays M or R instead, P2 would prefer A in both cases (dominant strat)

Introducing example

(L, B) is a well-defined pure-strategy NE but we do not really like it.

- ▷ Seems *implausible*
- ▷ It is **not sequentially rational**

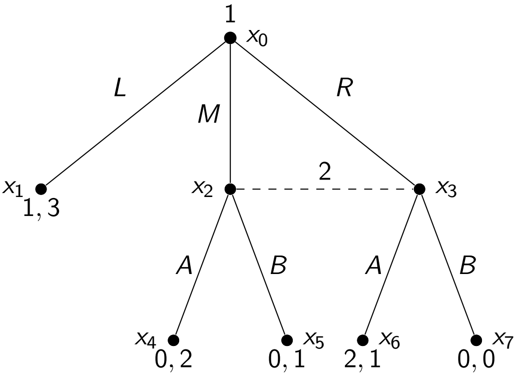
But it is a **dynamic game**.

- ▷ We have already seen that NE is not a *satisfactory* equilibrium concept for dynamic games.

That is why we introduced the notions of **subgames** and **subgame-perfect NE**.

- ▷ Let us apply this concept in the example!

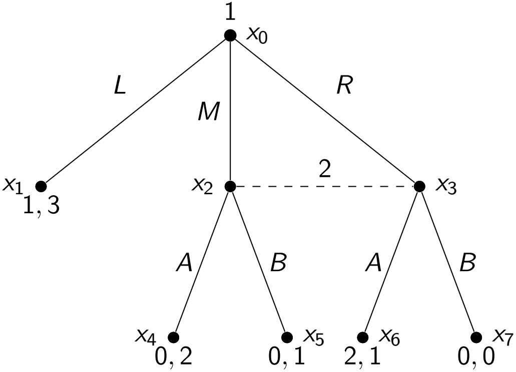
Introducing example



Extensive form:

- $N = \{1, 2\}$
- $A_1 = \{L, M, R\}$, $A_2 = \{A, B\}$
- $X_1 = \{x_0\}$, $X_2 = \{x_2, x_3\}$
- $I_1 = \{\{x_0\}\}$, $I_2 = \{\{x_2, x_3\}\}$
- $r = \{x_0\}$
- $T = \{x_1, x_4, x_5, x_6, x_7\}$

Introducing example



Extensive form:

- $N = \{1, 2\}$
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- $r = \{x_0\}$
- $T = \{x_1, x_4, x_5, x_6, x_7\}$

$\Rightarrow$ Only **one** subgame:

▷ The game itself!

Introducing example

Applying the **refinement** of subgame perfection **to this game**:

- ▷ Subgame-perfect NE $\Leftrightarrow$ Nash Equilibrium

It means that the subgame perfection refinement **has no bite** in this example.

- ▷ Applying it does not help removing the *unsatisfactory* Nash equilibrium (L, B).

We have to create another **refinement** to tackle this issue.

- ▷ It will be the **Perfect Bayesian Equilibrium**

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Preliminaries

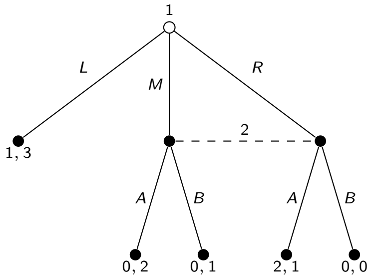
We would like to *get rid of* (L, B).

The idea
is to restore the notion of **non-credible threat** to *improper subgames*.

We
must then find a way for P2 to **distinguish** one node from another even when their information set does not allow it.

Beliefs are the key.

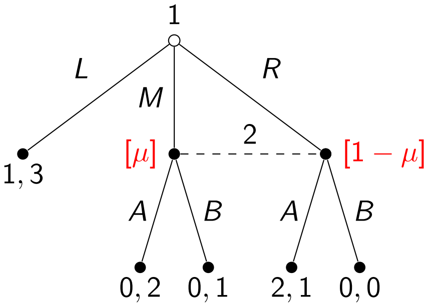
- We will allow P2 to **form beliefs on the probability** that M and R have been played by P1.
- And therefore to have a **strategy that depends on those beliefs**.



Preliminaries

For instance, assume that if P1 does not play L.

- P2 believes that M occurs with probability $\mu \in [0, 1]$ and R with probability $1 - \mu$.



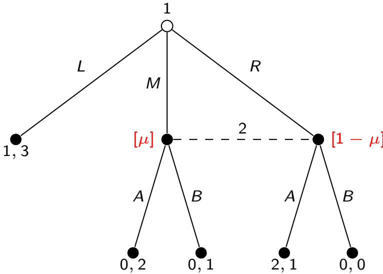
Preliminaries

P2's **expected payoff** when playing:

- *A*: $\mu \cdot 2 + (1-\mu) \cdot 1 = \mu + 1$
- *B*: $\mu \cdot 1 + (1-\mu) \cdot 0 = \mu$

For any $\mu \in [0, 1]$:

- P2 prefers to play A



Whatever P2's the belief,
B is **not a best-response anymore**.

This is enough to get rid of (L, B)

Preliminaries

Allowing P2 to have **beliefs on indistinguishable nodes** of their information set "solves" our problem of **non-credible threats**.

Natural questions:

1. Is it **reasonable** to assume that players have beliefs on indistinguishable nodes?
2. Where are those beliefs coming from?

Both questions will be answered by the new equilibrium concept.

- ▷ Key-feature: Beliefs will now be **considered part of the equilibrium**.
- ▷ Beliefs will emerge **endogenously** together with strategies.

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Beliefs and sequential rationality

Previous example: Useful to identify the **failure of subgame perfection** when the information set of a player is **not a singleton**.

Our goal is to define a setting in which we can say things like:

- "Player *i* is not **sequentially rational**"
- At every node where *i* plays, even if the information set **is not a singleton**

And then **apply this refinement** to Bayesian Nash equilibria of the game to remove *unreasonable* ones.

- As we did with subgame perfection: Take all the NE of the game and keep only those surviving subgame perfection (i.e. that are sequentially rational).

Methodology

We want to **refine** the concept of BNE.

To this end, we are going to impose

- ▷ **four requirements** on beliefs.

Finally, we will **impose those requirements** on BNE strategy profiles.

- ▷ and we will obtain a new equilibrium concept: **Perfect BNE**.

Decision nodes and information sets

Notation:

- X_i denote the set of player i's decision nodes
- H_i denote the set of player i's information sets
  - It is a partition of X_i

Example: Assume that in some extensive-form game player i's decision nodes are in $X_i = \{x_1, x_2, x_4, x_6\}$.

H_i can be any partition of X_i, for instance:

- $H_i = \{\{x_1\}, \{x_2\}, \{x_4\}, \{x_6\}\}$: All singletons
- $H_i = \{\{x_1, x_2\}, \{x_4\}, \{x_6\}\}$: x_1 and x_2 are not distinguishable
- $H_i = \{\{x_1, x_2, x_4\}, \{x_6\}\}$: only x_6 or "not x_6" is distinguishable
- $H_i = \{\{x_1, x_2, x_4, x_6\}\}$: nothing is distinguishable

System of beliefs

Definition: In an extensive-form game, a **system of beliefs** μ is a probability distribution over decision nodes within each information set.

Formally, for every player $i \in N$, every information set $h \in H_i$ and every of its decision node $x \in h$, $\mu(x) \in [0,1]$ is the probability that player i assigns to decision node x when player i moves to information set h.

Where $\sum_{x \in h} \mu(x) = 1$ for every $h \in H_i$, $i \in N$.

Example: Take $H_i = \{\{x_1, x_2, x_4\}, \{x_6\}\}$, $h_1 := \{\{x_1, x_2, x_4\}\}$ and $h_2 := \{\{x_6\}\}$ then the system of beliefs **may** assign:

- $\mu(x_1) = 2/3$, $\mu(x_2) = 1/6$, $\mu(x_4) = 1/6 \Rightarrow \sum_{x \in h_1} \mu(x) = 1$
- $\mu(x_6) = 1$

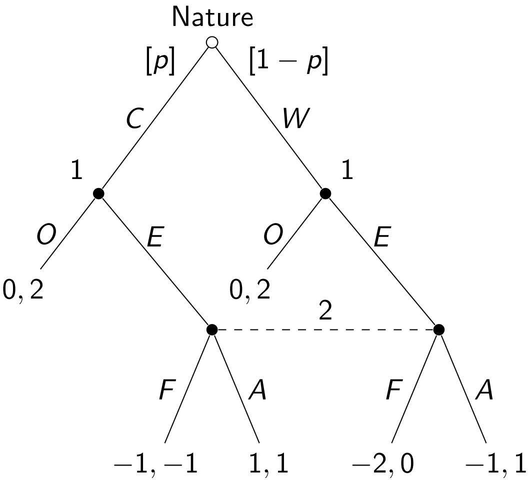
Beliefs: Requirement 1

Requirement 1: Every player has well-defined beliefs over their decision nodes at every of their information set (singleton or not)

That is, the game is endowed with a **complete system of beliefs**

Requirement 1: An Example

Consider the following game (Tadelis, 2013):

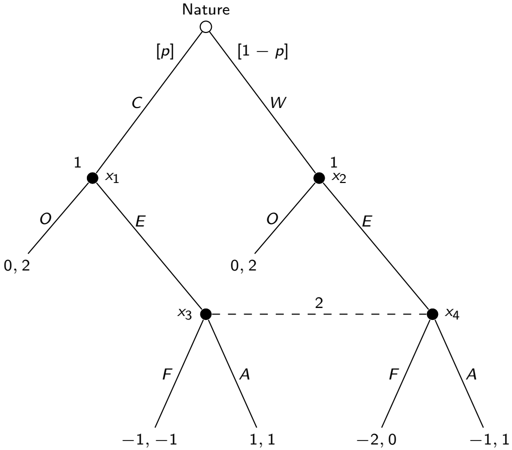


Requirement 1: An Example

A system of beliefs must assign a probability to x_1, x_2, x_3 and x_4.

We have

- $\mu(x_1) = \mu(x_2) = 1$
- $\mu(x_3) \in [0, 1]$
- $\mu(x_4) \in [0, 1]$
- $\mu(x_3) + \mu(x_4) = 1$



Beliefs: Where do they come from?

We have imposed a **system of beliefs** but how are they determined?

- ▷ Are they imposed by exogenous elements?
- ▷ Can players "choose" their beliefs?

We are going to allow for both in some way.

- ▷ **Exogenously:** Beliefs are partly determined by Nature.
- ▷ **Endogenously:** Beliefs are partly determined by **players' strategies**.

Beliefs: Consistency constraints

We impose some **consistency constraints** on beliefs.

- ▷ **Exogenously:** Beliefs must be consistent with Bayes' rule (we will be more specific later).
- ▷ **Endogenously:** Beliefs must be consistent with how we anticipate other players' strategies.

Reminder: Bayes' rule

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space

For any two events A, $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$ we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)}$$

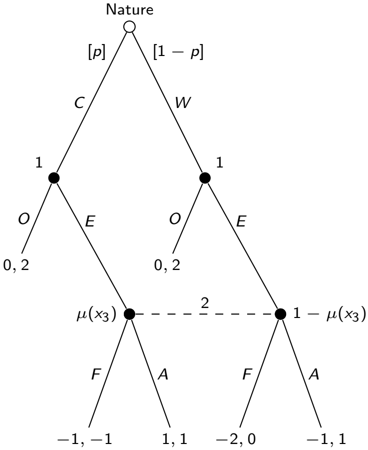
Beliefs: Consistency constraints

Go back to the example

- ▷ Constraints on $\mu(x_3)$?

Assume P1 plays EO, i.e.:

- ▷ When P1 is C: chooses E
- ▷ When P1 is W: chooses O



Beliefs: Consistency constraints

Go back to the example

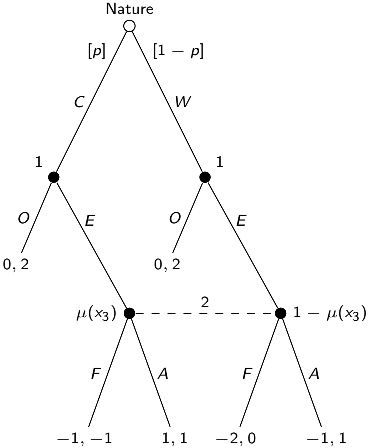
- ▷ Constraints on $\mu(x_3)$?

Assume P1 plays *EO*, i.e.

- ▷ When P1 is *C*: chooses *E*
- ▷ When P1 is *W*: chooses *O*

Belief consistency (endogenous):

- ▷ $\mu(x_3) = \mathbb{P}(\text{P1 is } C \mid E)$
- ▷ $1 - \mu(x_3) = \mathbb{P}(\text{P1 is } W \mid E)$

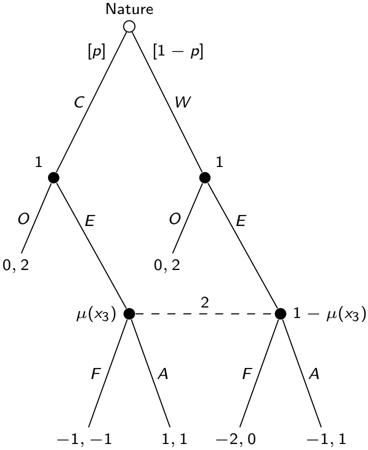


Beliefs: Consistency constraints

Therefore if P1 plays *EO* we must have:

- $\mu(x_3) = 1$.
- If P2 observes that the game reached this stage, it must be that P1 is **not** *W*.

P2's beliefs must be **consistent** with what P2 thinks P1 will play.



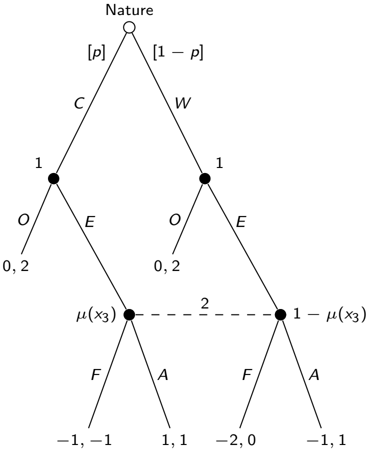
Beliefs: Consistency constraints

This also means that if $P1$ considers playing EO

- ▷ Anticipates that $\mu(x_3) = 1$.
- ▷ Can therefore anticipate that P2 **will play** A **following** E.

Then when P1 considers a **deviation** from EO.

- ▷ P1 could try to play E when W.
- ▷ P2 would **wrongly** believe that P1 is C and would play A.
- ▷ P1 therefore knows that this **deviation would not be profitable**



Beliefs: Consistency constraints

The above example illustrates an **endogenous consistency** requirement.

- When a player reaches a decision node for which the information set is not a singleton it must form a belief for each decision node that is **consistent** with the other players' strategies.

But it must also be **consistent with exogenous elements** such as Nature draw.

- We do not see it in the previous example
- See next slide

Beliefs: Consistency constraints

Consider the following case: P2 thinks that P1 of type

- C: chooses E with **probability** $\sigma_C \Leftrightarrow \mathbb{P}(E \mid C) = \sigma_C$
- W: chooses E with **probability** $\sigma_W \Leftrightarrow \mathbb{P}(E \mid W) = \sigma_W$

What should be $\mu(x_3)$?

- Assume P2 observes that P1 played E.
- Then, P2 must then infer how likely it is that P1 is C given that they played E.

Formally,

$$\mu(x_3) = \mathbb{P}(\text{Nature has chosen } C \mid \text{P1 played } E).$$

Beliefs: Consistency constraints

Using Bayes' rule we have that:

$$\mu(x_3) = \mathbb{P}(\text{Nature has chosen } C \mid \text{P1 played } E)$$

$$= \frac{\mathbb{P}(\text{Nature has chosen } C \;\; \text{AND} \;\; \text{P1 played } E)}{\mathbb{P}(\text{P1 played } E)}.$$

With lighter notations:

$$\mu(x_3) = \frac{\mathbb{P}(C \;\; \text{AND} \;\; E)}{\mathbb{P}(E)}.$$

Beliefs: Consistency constraints

Using **Bayes' rule** once again and the **law of total probability** we have:

$$\mu(x_3) = \frac{\mathbb{P}(C \;\; \text{AND} \;\; E)}{\mathbb{P}(E)}$$

$$= \frac{\mathbb{P}(E \mid C)\mathbb{P}(C)}{\mathbb{P}(E \mid C)\mathbb{P}(C) + \mathbb{P}(E \mid W)\mathbb{P}(W)}$$

$$= \frac{\sigma_C \cdot p}{\sigma_C \cdot p + \sigma_w \cdot (1-p)}$$

Reminder:

- $\mathbb{P}(A, B) = \mathbb{P}(A \mid B)\mathbb{P}(B)$ for any two A, $B \in \mathcal{F}$
- $\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$ where $(B_i)_{i=1}^m$ is a partition of $\mathcal{F}$

Beliefs: Consistency constraints

Notice that the belief

$$\mu(x_3) = \frac{\sigma_C \cdot p}{\sigma_C \cdot p + \sigma_w \cdot (1-p)},$$

depends both on

- ▷ P1's strategy: **Endogenous** consistency
- ▷ Nature's draw: **Exogenous** consistency

Rational players form their beliefs using **both of these elements**.

Notice also that if we set $\sigma_C = 1$ and $\sigma_W = 0$.

- ▷ P1 chooses E when C and O when W with **certainty**.
- ▷ Then $\mu(x_3) = \dfrac{1 \cdot p}{1 \cdot p + 0 \cdot (1-p)} = 1.$

Equilibrium path: On and off

We are almost ready to state our **second and third requirements** on beliefs.

But first, consider the following definition.

Definition (Tadelis, 2013): Let $\sigma^* = (\sigma_1^*, \dots \sigma_n^*)$ be a Bayesian Nash Equilibrium profile in a game of incomplete information. We say that an information set is **on the equilibrium path** if given σ^* and given the distribution of types, it is reached with **positive probability**.

By opposition, an information set is said to be **off the equilibrium path** if given σ^*, it is reached with **zero probability**.

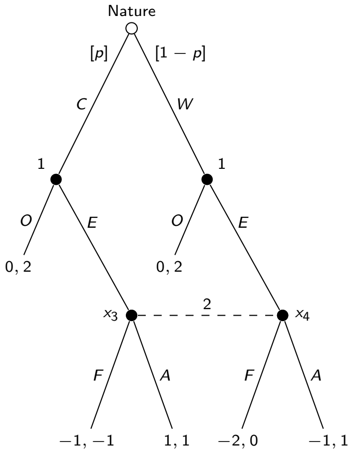
Equilibrium path: Example

Example:
Consider first that P1 chooses *EO*.

- With prob. p, P1 is C and plays E.
- With prob. $1 - p$, P1 is W and plays O.

$\Rightarrow$ The information set $h_1 = \{x_3, x_4\}$ is reached with positive probability p.

If *EO* were part of a BNE, we would say that h_1 is **on the equilibrium path** of this BNE.



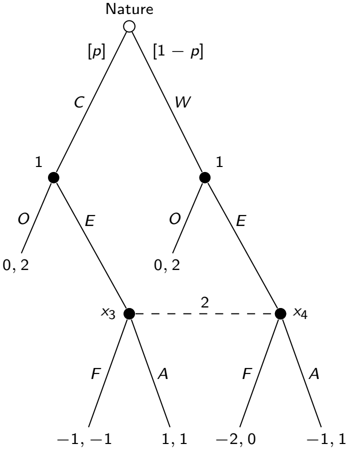
Equilibrium path: Example

Example: Consider now that P1 chooses *OO*

- With prob. p, P1 is C and plays O.
- With prob. $1 - p$, P1 is W and plays O.

$\Rightarrow$ The information set $h_1 = \{x_3, x_4\}$ is **never reached** with positive probability.

If *OO* were part of a BNE, we would say that h_1 is **off the equilibrium path** of this BNE.



Equilibrium path: On and off

Whether an information set is **on or off the equilibrium path** is not exogenous.

- ▷ It is determined by the players' actions.

We are now ready to state our **second and third requirements** on beliefs.

- ▷ One for beliefs **on** the equilibrium path.
- ▷ One for beliefs **off** the equilibrium path.

Beliefs: Requirement 2

Requirement 2 (Tadelis, 2013): For any BNE strategy profile σ^*, in all *information sets* that are **on** the equilibrium path, beliefs must be **consistent with Bayes' rule**.

That is, players must form their beliefs using both the

- **exogenous constraints** (nature);
- and the **endogenous constraints** (other players' strategies).

When Bayes is off path

What about information sets that are **off the equilibrium path**?

- Can't we **apply Bayes' rule** as well?
- Not always!

Recall that if P1's strategy is *OO*.

- Then $h_1 = \{x_3, x_4\}$ is **never reached**.

Assume that *P*2 **believes** that P1 plays *OO*.

- But surprisingly **observes** that $h_1 = \{x_3, x_4\}$ **is reached**!

Trying to apply Bayes' rule gives: $\mu(x_3) = \frac{0 \cdot p}{0 \cdot p + (1-p) \cdot 0} = \frac{0}{0}$!

When Bayes is off path

Clearly, applying Bayes' rule **fails** as $\mu(x_3) = \dfrac{0}{0}$ is **undefined**.

But you might wonder: **Why should we care?**

- ▷ It never happens at equilibrium, so why is that a problem?

When Bayes is off path

To see why, assume P2 believes that P1 plays *OO* so that Bayes' rule **does not apply to assign beliefs** to $h_1 = \{x_3, x_4\}$.

- ▷ P1 will compute their payoff with *OO* knowing that P2 will believe that h_1 is never reached.
- ▷ But if P1 wants to see if they could deviate from that and play *EO* for instance.
- ▷ Then, h_1 would be reached with positive probability.
- ▷ But as it is unexpected for P2, $\mu(x_3)$ is not defined by Bayes' rule.
- ▷ So that P1 is unable to know what will happen and to compute their payoff if they play E.

When Bayes is off path

Therefore, if $\mu(x_3)$ is undefined, we are unable to fully compute **how P1 could deviate** from OO.

That is why we will impose that there also exists beliefs over nodes of information sets that are **off the equilibrium path**.

We can now state our **third requirement**.

Beliefs: Requirement 3

Requirement 3 (Tadelis, 2013): At information sets that are **off the equilibrium path**, any belief can be assigned to which Bayes' rule does not apply.

In other words:

- ▷ If Bayes' rule can be applied: **Apply it!**
- ▷ Otherwise: $\mu(x)$ **can be anything in** $[0, 1]$ for x at an off equilibrium path information set.

Notice that there is room for some **arbitrary choices** here.

- ▷ We might obtain different solutions if we **choose different beliefs** off the equilibrium path!

Beliefs: Requirement 4

We can finally state our **fourth requirement** on beliefs.

Requirement 4 (Tadelis, 2013): Given their beliefs, players' strategies must be **sequentially rational**. That is, in every information set players will play a **best response** to their beliefs.

With this requirement, we restore the possibility to evaluate whether a move is **sequentially rational** or not at every information set, including those that contain more than one decision node.

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Perfect Bayesian Nash Equilibrium: Definition

Finally, we can define what is a **Perfect Bayesian Nash Equilibrium**.

Definition: A **Perfect Bayesian Nash Equilibrium** consists of a Bayesian Nash Equilibrium profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ together with a system of beliefs μ that satisfy Requirements 1,2,3 and 4.

In other words, a **PBNE** is a BNE such that players are **sequentially rational** at every information set.

PBNE: Beliefs and strategies

In BNE, beliefs were **purely exogenous**.

- ▷ Strategies **depended** on beliefs.
- ▷ But beliefs **were independent** of strategies.

The fundamental feature of PBNE is that beliefs and strategies are **both part of the equilibrium outcome**.

- ▷ Strategies **depend** on beliefs
- ▷ Beliefs **depend** $\begin{cases} \text{on Nature (what is given)} \\ \text{on strategies (what other players might do)} \end{cases}$

Beliefs emerge **endogenously**.

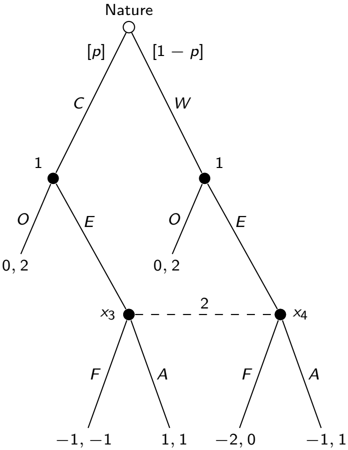
Perfect Bayesian Nash Equilibrium: An example

PBNE of this game with $p = 0.5$.

First, let us find the BNEs.

- ▷ We can compute the merged payoff matrix as follows.

1 \2	F	A
EE	$(-1,-2)\ ;-\frac{1}{2}$	$(1,-1)\ ;1$
EO	$(-1,0)\ ;\frac{1}{2}$	$(1,0)\ ;\frac{3}{2}$
OE	$(0,-2)\ ;1$	$(0,-1)\ ;\frac{3}{2}$
OO	$(0,0)\ ;2$	$(0,0)\ ;2$



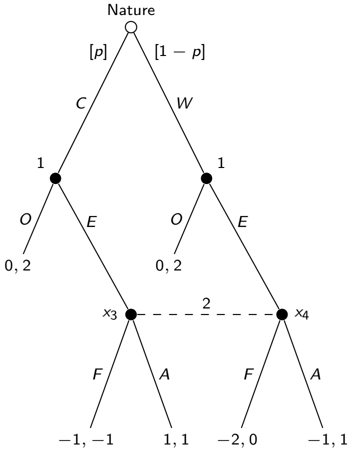
PBNE: An example

PBNE of this game with $p = 0.5$

First, let us find the BNEs

- ▷ We can compute the merged payoff matrix as follows

$1 \backslash 2$	F	A
EE	$(-1,-2)\ ;-\frac{1}{2}$	$(\underline{1},-1)\ ;\underline{1}$
EO	$(-1,\underline{0})\ ;\frac{1}{2}$	$(\underline{1},\underline{0})\ ;\underline{\frac{3}{2}}$
OE	$(\underline{0},-2)\ ;\underline{1}$	$(0,-1)\ ;\underline{\frac{3}{2}}$
OO	$(\underline{0},\underline{0})\ ;\underline{2}$	$(0,\underline{0})\ ;\underline{2}$



PBNE: An example

Two BNE: (OO, F) and (EO, A).

Consider (OO, F)**:**

- The information set $h_1 = \{x_3, x_4\}$ is **off the equilibrium path**.
- It means that $\mu(x_3)$ can be anything in $[0, 1]$ (Requirement 3).

However, assume that for some reason P2 observes $E \Rightarrow h_1$ is reached.

For any $\mu(x_3) \in [0, 1]$, P2's expected payoff is:

- $\mu(x_3) \cdot (-1) + (1 - \mu(x_3)) \cdot 0 = -\mu(x_3)$ if P2 plays F
- $\mu(x_3) \cdot 1 + (1 - \mu(x_3)) \cdot 1 = 1$ if P2 plays A

PBNE: An example

- ▷ $\mu(x_3) \cdot (-1) + (1 - \mu(x_3)) \cdot 0 = -\mu(x_3)$ if P2 plays F
- ▷ $\mu(x_3) \cdot 1 + (1 - \mu(x_3)) \cdot 1 = 1$ if P2 plays A

Then it is clear that P2 **will play** A for any value of $\mu(x_3)$.

In other words, (OO, F) is such that

- ▷ P2 is **not sequentially rational** (Requirement 4).

The BNE profile (OO, F) **does not survive** the PBNE refinement.

PBNE: An example

Consider now (EO, A)**:**

- ▷ The information set $h_1 = \{x_3, x_4\}$ is **on the equilibrium path**.
- ▷ Belief $\mu(x_3) = 1$ as only C chooses E.

P2 is then **certain** that observing E means that P1 is C.

So if P2 reaches h_1.

- ▷ **Best response is** A.

Are we done?

PBNE: An example

Are we done?

- ▷ Not yet, we also have to verify that EO **is a best response to** A **and belief** $\mu(x_3) = 1$.

Fix A and $\mu(x_3) = 1$.

1. P1 deviates to EE

- ▷ P2 would always believe that P1 is C.
- ▷ P2 would then always play A.

Not sequentially rational for P1.

- ▷ When reaching x_2, P1 knows that P2 will play A.
- ▷ Better playing O

PBNE: An example

Still fix A and $\mu(x_3) = 1$.

2. P1 deviates to OE

If P1 reaches x_1 and plays O.

- ▷ Not sequentially rational
- ▷ Would be better to play E so that P2 plays A

If P1 reaches x_2 and plays E.

- ▷ P2 will believe that P1 is C
- ▷ P2 will play A
- ▷ Not sequentially rational for P1 to play E at x_2

PBNE: An example

Still fix A and $\mu(x_3) = 1$.

3. P1 deviates to OO

If P1 reaches x_1 and plays O.

- ▷ Not sequentially rational
- ▷ Would be better to play E so that P2 plays A

PBNE: An example

Therefore:

- (EO, A) and $\mu(x_3) = 1$ is the **only PBNE** of the game.

- The other BNE profile is not sequentially rational according to our requirements.

PBNE: Introducing Example solution

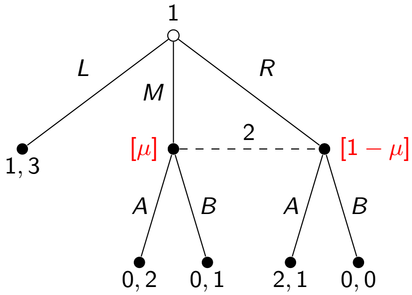
Let us go back to the introducing example.

Recall: Two BNE.

- (L, B)
- (R, A)

We have found that for any $\mu \in [0, 1]$.

- A is a dominant strategy if we reach P2's information set.



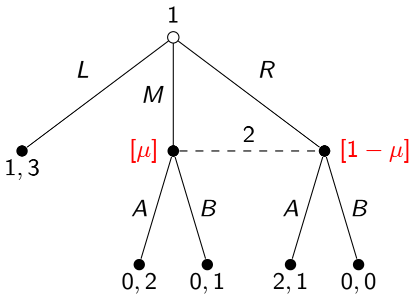
PBNE: Introducing Example solution

Consider (L, B).

- ▷ Information set is off the equilibrium path.
- ▷ Belief μ can be anything (requirement 3).
- ▷ But for any μ, A is a dominant strategy.

Then B is not sequentially rational.

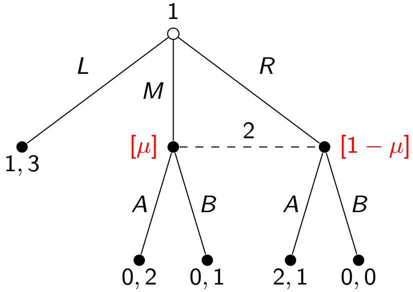
- ▷ (L, B) is not a PBNE.



PBNE: Introducing Example solution

Consider (R, A).

- ▷ Information set is on the equilibrium path.
- ▷ A is a dominant strat.
- ▷ P1 prefers R to M.
- ▷ Belief must be $\mu = 0$.



P1 can play:

- ▷ L and obtain 1.
- ▷ R and obtain 2 (better).

Therefore, (R, A) and $\mu = 0$ is a PBNE.

Other refinements

There exists **other refinements** of BNE.

For instance, **“sequential equilibrium”** is the most famous other one.

- ▷ Due to Kreps and Wilson (1982)

Sequential equilibrium is stronger than PBNE.

- ▷ Essentially a PBNE with **more requirements** on beliefs that are off the equilibrium path .
- ▷ Every sequential equilibrium is a PBNE, but the reverse does not hold.

In many applications, the two are equivalent.

- ▷ We will restrict to those cases

Table of Contents

Signaling games

A very important type of dynamic games of incomplete information:

- **Signaling games.**

They have some **distinguishable features**, which are, informally:

- P1 privately knows **payoff-relevant** information for P2
- No way to **certify** the information
- P1 will try to **signal** their information through their action
- Signaling is possible when actions are **credible signals**

Examples: Education, advertising, war games and sometimes even biological evolution!

Signaling games: Main Setting

The main setting for (two-player) signaling games is as follows:

1. Nature chooses a type. Only P1 learns it. But both P1 and P2's payoff depends on it;
2. P1 has at least as many actions as they have types (rich action space). Each action has some **cost**;
3. **Timing:** P1 plays first. P2 observes P1's action (but not type) and responds to it;
4. P2 updates their beliefs about P1's type thanks to their belief about P1's strategy and the observed action.

Signaling games: Classes of PBE

There are two important classes of PBE in signaling games.

1. **Pooling equilibria**
   - All types of P1 choose the same action, i.e., P1 *pools together* all their types in the same action.
   - P2 **cannot infer** anything about P1's type as their action is **non-revealing**.
   - P2 must then best respond using only **exogenous information** about P1's type.

Signaling games: Classes of PBE

The other class is

2. Separating equilibria

- ▷ Each of P1's type chooses a **different action**.
- ▷ P2 can **perfectly infer** P1's type from their action.
- ▷ P2 can then respond *as if* they were perfectly informed about P1's type.

In that case, we say that P1's action **reveals** their type

Signaling games: Pooling and Separating

Separating equilibria seem very powerful.

- ▷ P1 cannot provide **hard proof** of their type but can only send a **signal.**
- ▷ Yet, P2 becomes **perfectly informed.**
- ▷ All information is **revealed**!

Signaling games: Pooling and separating

Separating seems **too good to be true**. What could go wrong?

- Recall that both P1 and P2's payoff depend on P1's type
- Maybe they do not have **aligned interests**?
- When P2 learns the information, they may take an action that does not please P1
- P1 might then have an **incentive to lie/manipulate information** so that P2 responds in a way that P1 prefers
- But a **rational** P2 **will anticipate** this possibility
- So if P1 has an incentive to lie, **P2 should not believe that P1's action reveals their type**

Signaling games: Pooling and Separating

The **existence** of a separating equilibrium therefore **relies** on the **credibility of P1's signal.**

If both players' interests are aligned.

- ▷ P1's signal is **always credible.**

If both players' interests are **not aligned.**

- ▷ We must check that P1 **does not want to manipulate information.**
- ▷ This will crucially rely on whether **sending wrong signals is costly** enough for P1.

Separating and Pooling: Previous example

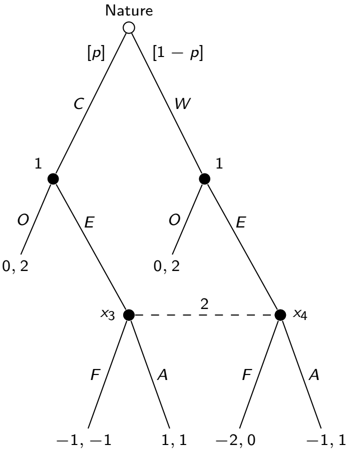
In the previous example:

The BNE: (OO, F) is a **pooling equilibrium**.

- ▷ Choice of $P1$ gives no information of their type.

The PBNE: (EO, A) is a **separating equilibrium**.

- ▷ If P1 plays E: P2 can perfectly infer that P1 is C.
- ▷ If P1 plays O: P2 can perfectly infer that P1 is W.



A famous game: Education as a signal

Famous signaling game: Education game **proposed by Spence (1973).**

Spence's idea is that education is a **signal of productivity.**

- ▷ Job recruiters **cannot observe** workers' productivity.
- ▷ An individual can spend time and effort studying to get a diploma.
- ▷ It is **less costly** to get the diploma for **more productive** individuals.

Additional assumption: Education has no effect on productivity.

- ▷ i.e., education is nonproductive, only a loss of time and efforts.
- ▷ seems unrealistic but not a problem, assuming education is productive would not change the result.

A famous game: Education as a signal

Big picture:

- Investing in education is **very** costly for individuals with a low level of productivity.
- We expect that **only** highly productive individuals invest in education.
- Therefore, **education signals productivity.**

The obvious problem to this reasoning is

- Low types must not be incentivized to get the diploma to pretend they are high types.

A famous game: Education as a signal

Setting (Tadelis, 2013, p.319):

Nature draws P1's type: With probability p, P1 is of type $t_1 = H$ (High); otherwise P1 is $t_1 = L$ (Low) with probability $1 - p$.

P1 plays first (after Nature):

- ▷ P1 is the future employee
- ▷ P1 can choose to study for an undergraduate degree U only or to continue studying to obtain a graduate degree D
- ▷ To obtain U: Individual cost is normalized to 0
- ▷ To obtain D: It costs $c_H = 2$ and $c_L = 5$ to type H and L, respectively

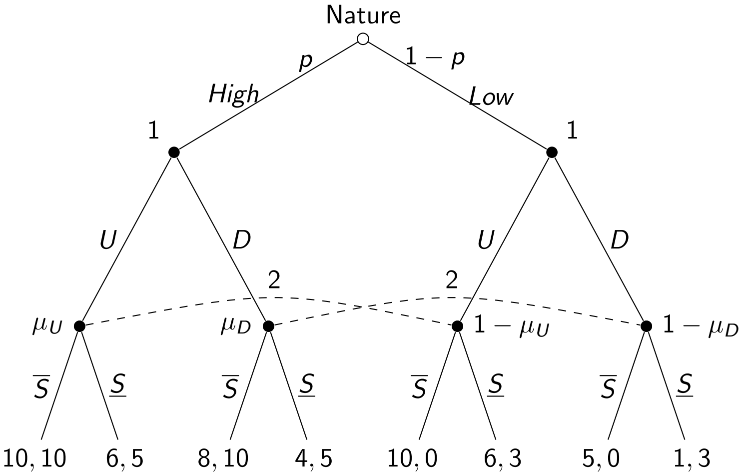
A famous game: Education as a signal

P2 observes $a_1 \in A_1 = \{U, D\}$ but not t_1

- P2 is the employer, plays after observing $a_1 \in A_1 = \{U, D\}$
- P2 must assign the employee to one of two possible tasks: $a_2 \in \{\underline{S}, \overline{S}\}$
- $\underline{S}$ is *less skilled* task than $\overline{S}$: The market wage for performing $\underline{S}$ is $\underline{w} = 6$ and the one for $\overline{S}$ is $\overline{w} = 10$
- P2's net profit depends on the following task-productivity assignment (independent of U and D):

	$\overline{S}$	$\underline{S}$
H	10	5
L	0	3

A famous game: Education as a signal



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A famous game: Education as a signal

We define a **system of beliefs**:

- μ_U: P2's belief that P1 is H after observing U.
- μ_D: P2's belief that P1 is H after observing D.

They will be determined both by

- Nature
- P1's strategy

A famous game: Education as a signal

First, let us find the BNEs when $p = \frac{1}{4}$.

There are **two pure-strategy BNE**:

- $(UU, \underline{SS})$ (Pooling)
- $(DU, \overline{S}\underline{S})$ (Separating)

The proof is left as an exercise

- See Tadelis (2013), p.322

A famous game: Education as a signal

Consider the separating equilibrium $(DU, \overline{S}\underline{S})$

- ▷ All information sets are on the equilibrium path

$$\mu_U = \mathbb{P}(H \mid U) = \frac{\mathbb{P}(U \mid H)\mathbb{P}(H)}{\mathbb{P}(U \mid H)\mathbb{P}(H) + \mathbb{P}(U \mid L)\mathbb{P}(L)}$$
$$= \frac{0 \cdot p}{0 \cdot p + 1 \cdot (1-p)}$$
$$= 0$$

When P2 observes U, they believe that P1 is L with certainty

A famous game: Education as a signal

Now for μ_D

$$\begin{aligned}\mu_D = \mathbb{P}(H \mid D) &= \frac{\mathbb{P}(D \mid H)\mathbb{P}(H)}{\mathbb{P}(D \mid H)\mathbb{P}(H) + \mathbb{P}(D \mid L)\mathbb{P}(L)} \\ &= \frac{1 \cdot p}{1 \cdot p + 0 \cdot (1 - p)} \\ &= 1\end{aligned}$$

When P2 observes D, they believe that P1 is H with certainty

A famous game: Education as a signal

For beliefs $\mu_U = 0$ and $\mu_D = 1$, it is clear that

- ▷ $\overline{S}$ is a BR to D $(10 > 5)$
- ▷ $\underline{S}$ is a BR to U $(3 > 0)$

For beliefs $\mu_U = 0$, $\mu_D = 1$ and $\overline{S}\underline{S}$:

- ▷ P1 of type H: Prefers D to U $(8 > 6)$
- ▷ P1 of type L: Prefers U to D $(6 > 5)$

Therefore $(DU, \overline{S}\underline{S})$ is a (separating) PBNE

A famous game: Education as a signal

Consider the pooling equilibrium $(UU, \underline{SS})$

- The information set for nodes after D is off the equilibrium path

For the one on the equilibrium path

- $\mu_U = p = \frac{1}{4}$
- that is, this belief for P2 is only constituted by the **exogenous information**
- Because P1 **reveals nothing** by playing U for each of their type

A famous game: Education as a signal

Consider the pooling equilibrium $(UU, \underline{SS})$

- Information set for nodes after D: off the equilibrium path
- Information set for nodes after U: on the equilibrium path

For the one **on** the equilibrium path

- $\mu_U = p = \frac{1}{4}$
- that is, this belief for P2 is only constituted by the **exogenous information**
- Because P1 **reveals nothing** by playing U for each of their type

A famous game: Education as a signal

For the one **off** the equilibrium path:

- μ_D can be anything between $[0, 1]$.

Let us compute P2's best response to observing D for belief $\mu_D \in [0, 1]$

- Playing $\overline{S}$: $10\mu_D + 0(1 - \mu_D) = 10\mu_D$
- Playing $\underline{S}$: $5\mu_D + 3(1 - \mu_D) = 2\mu_D + 3$

Therefore P1 prefers $\begin{cases} \overline{S} & \text{if } \mu_D \geq \frac{3}{8} \\ \underline{S} & \text{if } \mu_D \leq \frac{3}{8} \end{cases}$

A famous game: Education as a signal

Therefore, for $\underline{S}$ to be a BR for P2 it must be the case that

$$\mu_D \leq \frac{3}{8}$$

This means that $(UU, \underline{SS})$ is a PBNE only if beliefs off the equilibrium path $\mu_D \in [0, \frac{3}{8}]$